Basic Engineering

Boolean Algebra and Logic Gates

F Hamer, M Lavelle & D McMullan

The aim of this document is to provide a short, self assessment programme for students who wish to understand the basic techniques of logic gates.

 Email: chamer, mlavelle, dmcmullan@plymouth.ac.uk

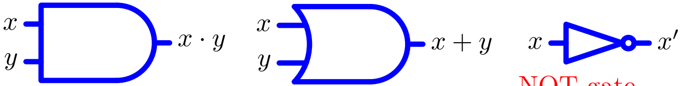
Last Revision Date: August 31, 2006 Version 1.0

Table of Contents

The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Logic Gates (Introduction)

The package **Truth Tables and Boolean Algebra** set out the basic principles of logic. Any Boolean algebra operation can be associated with an electronic circuit in which the inputs and outputs represent the statements of Boolean algebra. Although these circuits may be complex, they may all be constructed from three basic devices. These are the AND gate, the OR gate and the NOT gate.



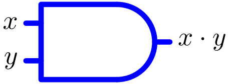
AND gate OR gate NOT gate

In the case of logic gates, a different **notation** is used:

$x \wedge y$, the logical AND operation, is replaced by $x \cdot y$, or xy.
$x \vee y$, the logical OR operation, is replaced by $x + y$.
$\neg x$, the logical NEGATION operation, is replaced by x' or $\overline{x}$.

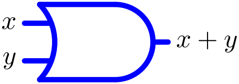
The truth value TRUE is written as 1 (and corresponds to a high voltage), and FALSE is written as 0 (low voltage).

2. Truth Tables



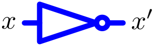
x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

Summary of AND gate



x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

Summary of OR gate



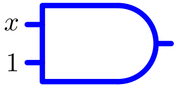
x	x'
0	1
1	0

Summary of NOT gate

3. Basic Rules of Boolean Algebra

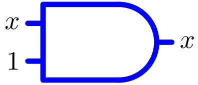
The basic rules for simplifying and combining logic gates are called Boolean algebra in honour of George Boole (1815 − 1864) who was a self-educated English mathematician who developed many of the key ideas. The following set of exercises will allow you to rediscover the basic rules:

Example 1

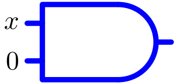


Consider the AND gate where one of the inputs is 1. By using the truth table, investigate the possible outputs and hence simplify the expression $x \cdot 1$.

Solution From the truth table for AND, we see that if x is 1 then $1 \cdot 1 = 1$, while if x is 0 then $0 \cdot 1 = 0$. This can be summarised in the rule that $x \cdot 1 = x$, i.e.,

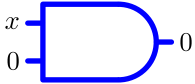


Example 2



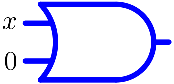
Consider the AND gate where one of the inputs is 0. By using the truth table, investigate the possible outputs and hence simplify the expression $x \cdot 0$.

Solution From the truth table for AND, we see that if x is 1 then $1 \cdot 0 = 0$, while if x is 0 then $0 \cdot 0 = 0$. This can be summarised in the rule that $x \cdot 0 = 0$

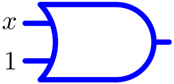


Exercise 1. (Click on the green letters for the solutions.) Obtain the rules for simplifying the logical expressions

(a) $x + 0$ which corresponds to the logic gate

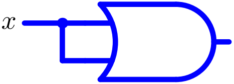


(b) $x + 1$ which corresponds to the logic gate

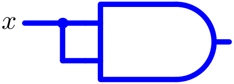


Exercise 2. (Click on the green letters for the solutions.) Obtain the rules for simplifying the logical expressions:

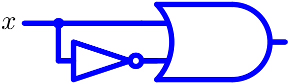
(a) $x + x$ which corresponds to the logic gate

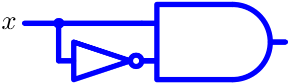


(b) $x \cdot x$ which corresponds to the logic gate

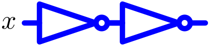


Exercise 3. (Click on the green letters for the solutions.) Obtain the rules for simplifying the logical expressions:

(a) $x + x'$ which corresponds to the logic gate 

(b) $x \cdot x'$ which corresponds to the logic gate 

Quiz Simplify the logical expression $(x')'$ represented by the following circuit diagram.



(a) x (b) x' (c) 1 (d) 0

Exercise 4. (Click on the green letters for the solutions.) Investigate the relationship between the following circuits. Summarise your conclusions using Boolean expressions for the circuits.

(a)

(b)

The important relations developed in the above exercise are called De Morgan's theorems and are widely used in simplifying circuits. These correspond to rules (8a) and (8b) in the table of Boolean identities on the next page.

4. Boolean Algebra

(1a)	$x \cdot y$	$=$	$y \cdot x$
(1b)	$x + y$	$=$	$y + x$
(2a)	$x \cdot (y \cdot z)$	$=$	$(x \cdot y) \cdot z$
(2b)	$x + (y + z)$	$=$	$(x + y) + z$
(3a)	$x \cdot (y + z)$	$=$	$(x \cdot y) + (x \cdot z)$
(3b)	$x + (y \cdot z)$	$=$	$(x + y) \cdot (x + z)$
(4a)	$x \cdot x$	$=$	x
(4b)	$x + x$	$=$	x
(5a)	$x \cdot (x + y)$	$=$	x
(5b)	$x + (x \cdot y)$	$=$	x
(6a)	$x \cdot x'$	$=$	0
(6b)	$x + x'$	$=$	1
(7)	$(x')'$	$=$	x
(8a)	$(x \cdot y)'$	$=$	$x' + y'$
(8b)	$(x + y)'$	$=$	$x' \cdot y'$

These rules are a direct translation into the notation of logic gates of the rules derived in the package **Truth Tables and Boolean Algebra**. We have seen that they can all be checked by investigating the corresponding truth tables. Alternatively, some of these rules can be derived from simpler identities derived in this package.

Example 3 Show how rule (5a) can be derived from the basic identities derived earlier.

Solution

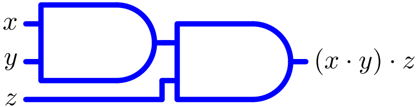
$$\begin{aligned} x\cdot(x+y) &= x\cdot x + x\cdot y \quad \text{using (3a)} \\ &= x + x\cdot y \quad \text{using (4a)} \\ &= x\cdot(1+y) \quad \text{using (3a)} \\ &= x\cdot 1 \quad \text{using Exercise 1} \\ &= x \quad \text{as required.} \end{aligned}$$

Exercise 5. (Click on the green letter for the solution.)

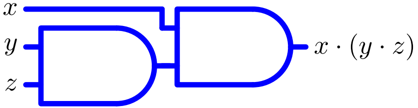
(a) Show how rule (5b) can be derived in a similar fashion.

The examples above have all involved at most two inputs. However, logic gates can be put together to join an arbitrary number of inputs. The Boolean algebra rules of the table are essential to understand when these circuits are equivalent and how they may be simplified.

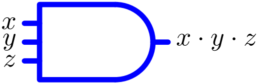
Example 4 Let us consider the circuits which combine three inputs via AND gates. Two different ways of combining them are



and



However, rule (2a) states that these gates are equivalent. The order of taking AND gates is not important. This is sometimes drawn as a three (or more!) input AND gate

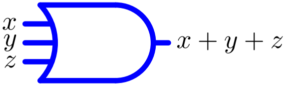


but really this just means repeated use of AND gates as shown above.

Exercise 6. (Click on the green letter for the solution.)

(a) Show two different ways of combining three inputs via OR gates and explain why they are equivalent.

This equivalence is summarised as a three (or more!) input OR gate



this just means repeated use of OR gates as shown in the exercise.

5. Final Quiz

Begin Quiz

1. Select the Boolean expression that is *not* equivalent to $x \cdot x + x \cdot x'$

(a) $x \cdot (x + x')$ (b) $(x + x') \cdot x$ (c) x' (d) x

2. Select the expression which is equivalent to $x \cdot y + x \cdot y \cdot z$

(a) $x \cdot y$ (b) $x \cdot z$ (c) $y \cdot z$ (d) $x \cdot y \cdot z$

3. Select the expression which is equivalent to $(x + y) \cdot (x + y')$

(a) y (b) y' (c) x (d) x'

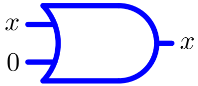
4. Select the expression that is *not* equivalent to $x \cdot (x' + y) + y$

(a) $x \cdot x' + y \cdot (1 + x)$ (b) $0 + x \cdot y + y$ (c) $x \cdot y$ (d) y

End Quiz Score: Correct

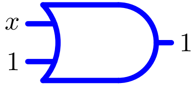
Solutions to Exercises

Exercise 1(a) From the truth table for OR, we see that if x is 1 then $1 + 0 = 1$, while if x is 0 then $0 + 0 = 0$. This can be summarised in the rule that $x + 0 = x$



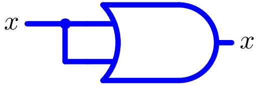
Click on the green square to return □

Exercise 1(b) From the truth table for OR we see that if x is 1 then $1 + 1 = 1$, while if x is 0 then $0 + 1 = 1$. This can be summarised in the rule that $x + 1 = 1$



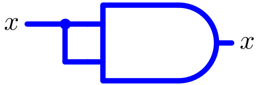
Click on the green square to return □

Exercise 2(a) From the truth table for OR, we see that if x is 1 then $x + x = 1 + 1 = 1$, while if x is 0 then $x + x = 0 + 0 = 0$. This can be summarised in the rule that $x + x = x$



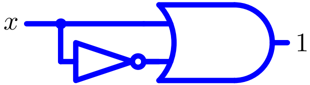
Click on the green square to return □

Exercise 2(b) From the truth table for AND, we see that if x is 1 then $x \cdot x = 1 \cdot 1 = 1$, while if x is 0 then $x \cdot x = 0 \cdot 0 = 0$. This can be summarised in the rule that $x \cdot x = x$



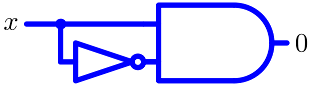
Click on the green square to return □

Exercise 3(a) From the truth table for OR, we see that if x is 1 then $x + x' = 1 + 0 = 1$, while if x is 0 then $x + x' = 0 + 1 = 1$. This can be summarised in the rule that $x + x' = 1$



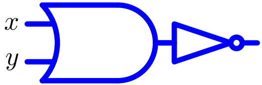
Click on the green square to return □

Exercise 3(b) From the truth table for AND, we see that if x is 1 then $x \cdot x' = 1 \cdot 0 = 0$, while if x is 0 then $x \cdot x' = 0 \cdot 1 = 0$. This can be summarised in the rule that $x \cdot x' = 0$

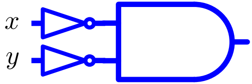


Click on the green square to return □

Exercise 4(a) The truth tables are:

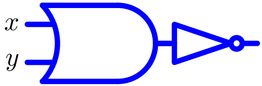
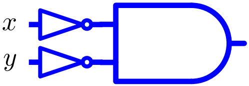


x	y	$x+y$	$(x+y)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0



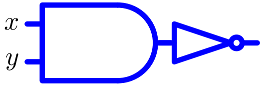
x	y	x'	y'	$x'\cdot y'$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

From these we deduce the identity

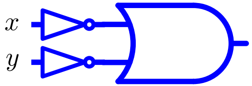
 $(x+y)' =$  $x'\cdot y'$

Click on the green square to return □

Exercise 4(b) The truth tables are:

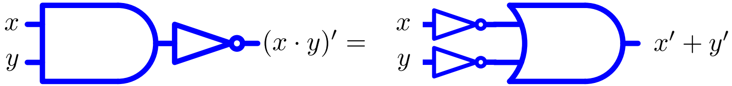


x	y	$x \cdot y$	$(x \cdot y)'$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0



x	y	x'	y'	$x' + y'$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

From these we deduce the identity



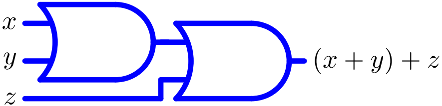
Click on the green square to return □

Exercise 5(a)

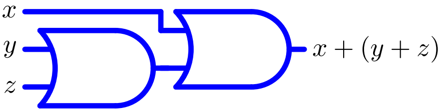
$$\begin{aligned} x + x \cdot y &= x \cdot (1 + y) \quad \text{using (3a)} \\ &= x \cdot 1 \quad \text{using Exercise 1} \\ &= x \quad \text{as required.} \end{aligned}$$

□

Exercise 6(a) Two different ways of combining them are



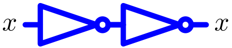
and



However, rule (2b) states that these gates are equivalent. The order of taking OR gates is not important. □

Solutions to Quizzes

Solution to Quiz: From the truth table for NOT we see that if x is 1 then $(x')' = (1')' = (0)' = 1$, while if x is 0 then $(x')' = (0')' = (1)' = 0$. This can be summarised in the rule that $(x')' = x$



End Quiz