Basic Engineering

AC Systems and Phasors

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The aim of this package is to provide a short self assessment programme for students who want to understand AC systems and the basic idea of phasors.

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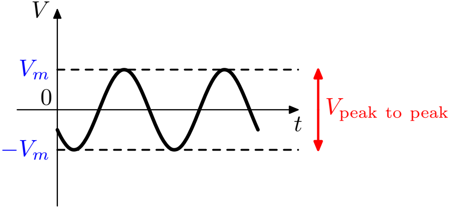
The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction (AC Systems)

This package discusses AC (alternating current) systems. We restrict ourselves to sinusoidal oscillations (see the package on **Waves**). The diagram shows the voltage measured at a fixed point as a function of time:

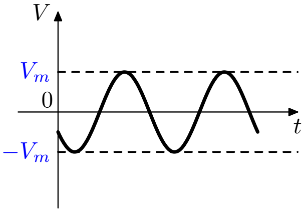


This function can be described by

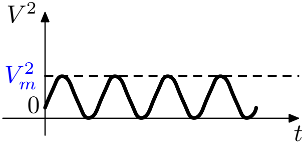
$$V = V_m \cos(2\pi f t + \phi)$$

Here V_m is the maximum voltage ($-V_m$ is the minimum); f is the frequency and ϕ is a phase. V_m is also called the **peak voltage** or **amplitude**. The peak to peak voltage is $2V_m$.

Since AC voltage varies with time between $\pm V_m$ we need a way to describe its *typical value*. The average voltage is zero (the positive and negative voltages cancel out). But the square of the voltage is, see the diagrams below, always positive:



Voltage (*volts*) against time



Voltage squared (*volts squared*) against time

Now take the square root of the average of V^2. This defines the root mean square or RMS voltage (units are $\sqrt{\text{volts}^2} = \text{volts}$). It can be shown that

$$V_{\text{RMS}} = \frac{V_m}{\sqrt{2}}$$

Example 1 An important example is $V_m = 340\,\text{V}$ and $f = 50\,\text{Hz}$. This means that every second the voltage supply undergoes 50 cycles. Its period is $T = 1/f = 1/50 = 0.02\,\text{s}$. This is the voltage and frequency of domestic electricity supply in the UK.

Since in domestic electricity supply $V_m = 340\,\text{V}$ the RMS voltage is $V_{\text{RMS}} = 340/\sqrt{2} = 240\text{V}$ to the nearest volt.

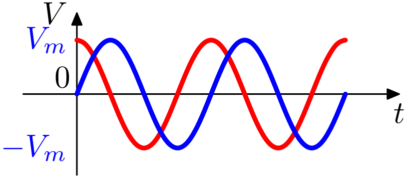
Note that AC *voltmeters display the value of the RMS voltage*. The RMS values are also used for comparisons with DC systems.

Quiz Electricity in transmission lines is typically transmitted at higher voltages to reduce power loss. If the RMS voltage in a SuperGrid transmission line is $400\,\text{kV}$ what is the peak voltage in the line?

(a) $283\,\text{kV}$ (b) $340\,\text{V}$ (c) $566\,\text{kV}$ (d) $400\,\text{kV}$

2. From Phases to Phasors

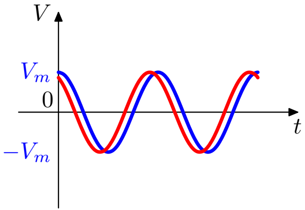
Example 2 Consider the two AC voltages below:



The peaks or troughs of the red curve are measured at earlier times than their blue equivalents. The red curve is ahead of the blue curve by one quarter of a cycle ($1\text{cycle} = 360°$ or 2π). As the frequencies are the same, this phase difference stays constant. Here is some **notation**:

- the **red curve leads** the blue curve by $90°$.
- it has a **positive phase shift** of $90°$ with respect to the blue curve.
- the **blue curve lags** the red curve by $90°$.
- It has a **negative phase shift** of $90°$ with respect to the red curve.

Quiz Which of the statements about the curves below is correct?



(a) The blue curve leads the red one by $30°$.

(b) The red curve has a negative phase shift of $30°$ with respect to the blue curve.

(c) The red curve lags the blue one by $30°$.

(d) The blue curve has a negative phase shift of $30°$ with respect to the red curve.

Exercise 1. In each case below sketch two AC voltages, V_1 and V_2, which have the same frequency against time for the given relative phases. In your sketch *assume* that V_1 has twice as large a peak voltage as V_2.

Hint: draw, say, V_1 first and then add V_2 with the requested phase shift and smaller amplitude.

(a) V_1 **lags** V_2 by $180°$.

(b) V_1 **leads** V_2 by $90°$.

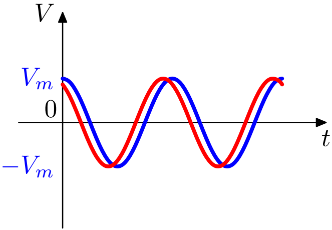
(c) V_2 **lags** V_1 by $90°$.

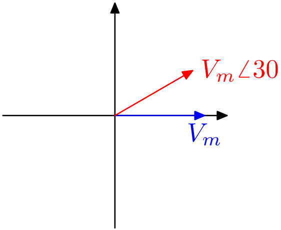
(d) V_1 **lags** V_2 by $45°$.

(e) V_1 has a **negative phase shift** with respect to V_2 of $60°$.

(f) V_1 has a **positive phase shift** with respect to V_2 of $90°$.

Example 3 **Phasors** are an efficient method of analysing AC circuits when the frequencies are the same. (The amplitudes do not need to be.) Let us draw the example of the last quiz in phasor form:

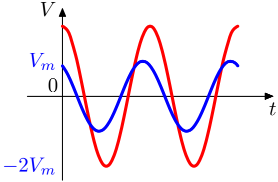




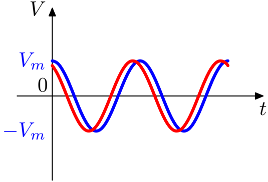
In the **phasor diagram**: everything is defined relative to the **reference phasor** (which is always chosen to point to the right). Here we have chosen the blue voltage as our reference.

- The two voltages have the same amplitude. Therefore the arrow of the red phasor has the same **length** as the reference phasor.
- The red voltage *leads* the blue voltage by $30°$: the red phasor is rotated *anti-clockwise* by a **relative angle** of $30°$ (written $V_m\angle 30°$).

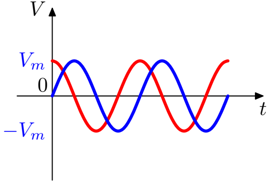
Exercise 2. Re-express each case below as a phasor diagram.

(a) 

Relative angle 30°. Use blue curve as reference.

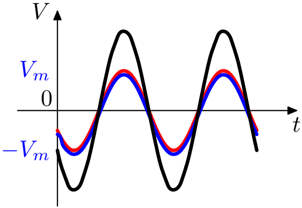
(b) 

Relative angle 30°. Use red curve as reference.

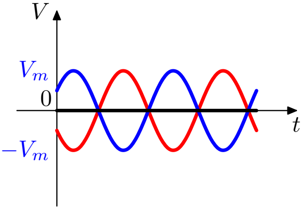
(c) 

Read off relative angle from graph. Use blue curve as reference.

The result of adding two AC quantities depends on their **relative phase**. This is a typical wave interference phenomenon (see the package on **Waves**). Below two voltages (red and blue lines) are added and the resulting voltage (black line) is shown.



Adding in phase
(*adds*)



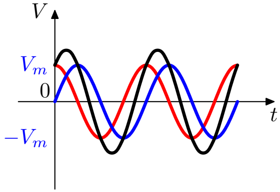
Adding out of phase
(*cancels*)

> If all signals have one, fixed frequency, adding them produces a signal with the same frequency but a different amplitude and a phase shift. We will not prove this result here.

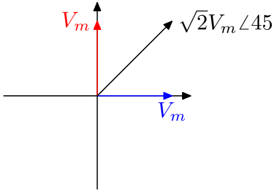
3. Adding Phasors (Introduction)

Phasors can be used to combine AC voltages or currents.

Example 4 Below two voltages (red and blue curves), which are out of phase by 90° are added. The result (black curve) has the same frequency but a larger amplitude and its peaks lie exactly between the other two.

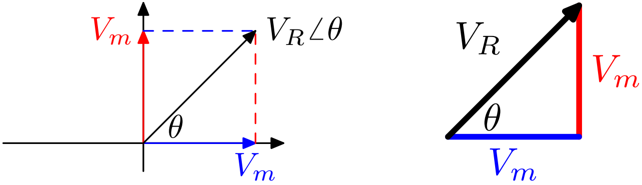


Adding two AC voltages



Same result using phasors

This result can be obtained directly by adding the two voltages in the phasor diagram. They are added in the same way that vectors are added. This is calculated in detail on the next page.



From the right angled triangle, the **angle** θ is given by

$$\tan(\theta) = \frac{V_m}{V_m} = 1\,, \qquad \Rightarrow \theta = \tan^{-1}(1) = 45^\circ\,.$$

while the amplitude of the resultant voltage, V_R, is the length of the hypotenuse: by Pythagoras's theorem

$$V_R^2 = V_m^2 + V_m^2\,, \qquad \Rightarrow \quad V_R = \sqrt{2V_m^2} = \sqrt{2}V_m$$

The result is thus: $V_m\angle 0^\circ + V_m\angle 90^\circ = \sqrt{2}V_m\angle 45^\circ$.

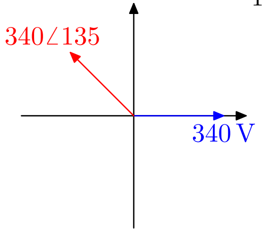
Quiz What is the result of adding $3\angle 0^\circ + 4\angle 90^\circ$ (volts)?

(a) $7\angle 90^\circ$ V (b)$5\angle 53^\circ$ V (c) $5\angle 1^\circ$ V (d) $12\angle 60^\circ$ V

Example 5

Here we add: $340\angle 0 + 340\angle 135$ (volts)

N.B. the red phasor points left at $180 - 135 = 45°$ above the horizontal.



Calculate the components along the axes. These are **horizontally**:

$$340 - 340\cos(45) = 100\,\mathrm{V}$$

and **vertically**:

$$0 + 340\sin(45) = 240\,\mathrm{V}$$

Thus (by Pythagoras's theorem) the **resultant peak voltage** is

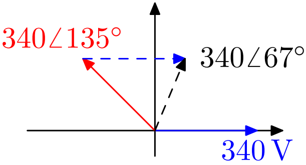
$$V_R = \sqrt{100^2 + 240^2} = 260\,\mathrm{V}$$

and the **phase** is

$$\theta = \tan^{-1}\left(\frac{240}{100}\right) = 67°\,.$$

This result is shown on the next page.

The result, $340\angle 0^\circ + 340\angle 135^\circ = 260\angle 67^\circ$, is shown below:



The arrows show the geometrical way to add phasors. Phasors pointing in opposite directions tend to cancel each other.

EXERCISE 3. Here are some exercises on adding phasors. In each case draw a phasor diagram and thus calculate the sum.

(a) $12\angle 0^\circ + 5\angle 90^\circ$ (volts) (b) $3\angle 0^\circ + 4\angle -90^\circ$ (volts)

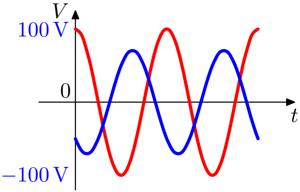
(c) $1\angle 0^\circ + 2\angle 60^\circ$ (amperes) (d) $2\angle 0^\circ + 2\angle 30^\circ$ (volts)

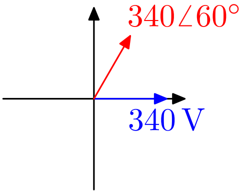
Quiz What is the result of $12\angle 0^\circ + 5\angle 180^\circ$ (volts)?

(a) $7\angle 180^\circ$ V (b) $17\angle 90^\circ$ V (c) $13\angle 85^\circ$ V (d) $7\angle 0^\circ$ V

4. Final Quiz

Begin Quiz





1. In the graph, what is the RMS voltage of the red curve?
(a) 141 V (b) 282 V (c) 200 V (d) 71 V

2. Which statement about the blue curve below is *wrong*?
(a) It has a positive phase shift (b) It leads the red curve
(c) It has a negative phase shift (d) The red curve lags it.

3. What would be the result of adding the phasors in the diagram?
(a) $680\angle 60^\circ$ V (b) $524\angle 24^\circ$ V (c) $340\angle 30^\circ$ V (d) $589\angle 30^\circ$ V

End Quiz Score: Correct

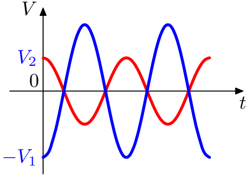
Solutions to Exercises

Exercise 1(a)

Since the AC voltage V_1 lags V_2 by $180°$ and its peak is twice as large as the V_2 peak, the voltage V_1 is given by

$$V_1 = 2V_2 \cos(2\pi f t - 180)$$

and the diagram for the voltages is



A phase difference of $180°$ means that the two voltages are completely out of phase.

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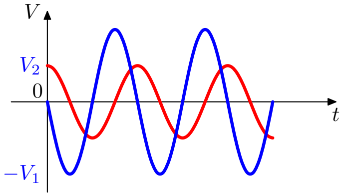
Exercise 1(b)
The AC voltage V_1 leads V_2 by $90°$ and its peak is twice as large as the V_2 peak therefore the voltage V_1 is

$$V_1 = 2V_2 \cos(2\pi ft + 90)$$

and the diagram is



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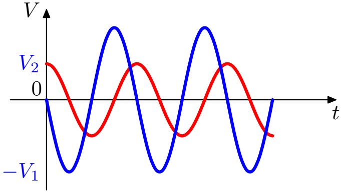
Exercise 1(c)

If V_2 **lags** V_1 by $90°$, then V_1 **leads** V_2 by $90°$. This is therefore just a different way of expressing the question of part (**b**) of the present exercise.



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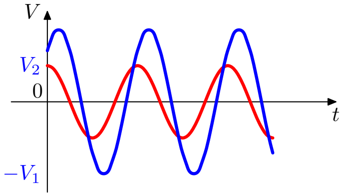
Exercise 1(d)

The AC voltage V_1 lags V_2 by $45°$ and its peak is twice as large as the V_2 peak, therefore the voltage V_1 is

$$V_1 = 2V_2 \cos(2\pi f t - 45)$$

and the diagram is



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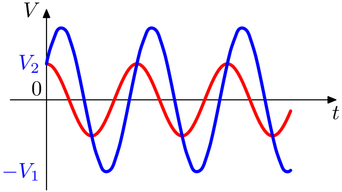
Exercise 1(e)

The AC voltage V_1 has a **negative phase shift** with respect to V_2 of $60°$ and its peak is twice as large as the V_2 peak therefore the voltage V_1 is given by

$$V_1 = 2V_2 \cos(2\pi f t - 60)$$

and the voltages are



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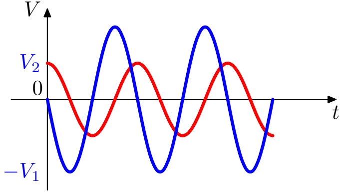
Exercise 1(f)

If V_1 has a **positive phase shift** with respect to V_2 of $90°$, then V_1 **leads** V_2 by $90°$ and therefore this is just a different way of expressing the question of part (**b**).



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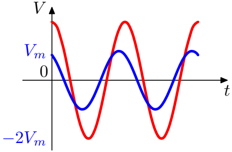
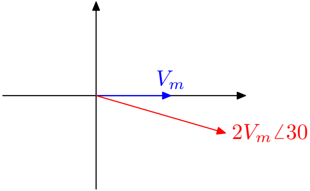
Exercise 2(a)

The blue reference voltage leads the red voltage by 30° and the peak voltages are related by $V_{\text{red}} = 2V_{\text{blue}}$.



Therefore in the phasor diagram the red phasor is rotated *clockwise* by 30° and the arrow of the red phasor is twice as long as the blue reference phasor:



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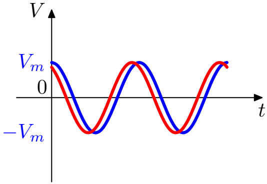
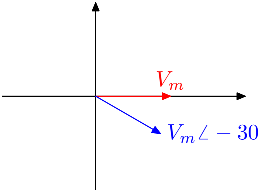
□

Exercise 2(b)

The blue voltage lags the red reference voltage by 30° and the peak voltages are the same $V_{\text{red}} = V_{\text{blue}}$.



Therefore in the phasor diagram the blue phasor is rotated *clockwise* by 30° relative to the red one and they have the same length:



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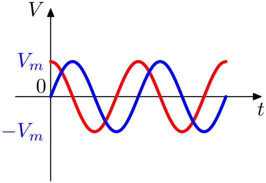
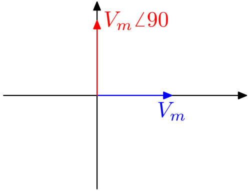
Exercise 2(c)

From the graph, the red voltage leads the blue reference voltage by 90° while the peak voltages are equal.



Therefore in the phasor diagram the red phasor is rotated *anti-clockwise* by 90° relative to the blue one and the length are the same:



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Exercise 3(a)

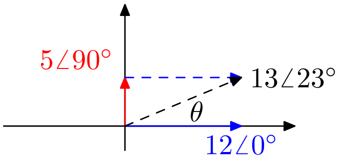
The sum of two phasors $12\angle 0^\circ\mathrm{V}$ and $5\angle 90^\circ$ V is the sum of two vectors $\mathbf{V}_R = \mathbf{V}_1 = (12, 0)\,, \mathbf{V}_2 = (0, 5)$ (see the package **Introduction to Vectors**). The length of the resulting vector $\mathbf{V}_1 + \mathbf{V}_2 = (12, 5)$ is

$$V_R \quad = \quad \sqrt{12^2 + 5^2} = \sqrt{169} = 13\,\mathrm{V}\,,$$

and its angle relative to $\mathbf{V}_1$ is

$$\tan(\theta) = \frac{5}{12} \approx 0.42\,, \qquad \Rightarrow \theta = \tan^{-1}(0.42) \approx 23^\circ\,.$$

The result $12\angle 0^\circ\mathrm{V} + 5\angle 90^\circ\mathrm{V} = 13\angle 23^\circ\mathrm{V}$ is drawn below



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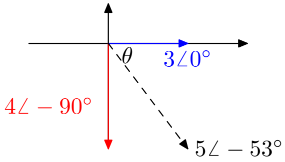
Exercise 3(b) The sum of two phasors $3\angle 0°\mathrm{V}$ and $4\angle -90°\ \mathrm{V}$ is a sum of two vectors $\mathbf{V}_1 = (3, 0)\,, \quad \mathbf{V}_2 = (0, -4)\,.$ The length and the relative angle of the resulting voltage $\mathbf{V}_R = \mathbf{V}_1 + \mathbf{V}_2 = (3, -4)$ are

$$V_R \;=\; \sqrt{3^2+4^2} = \sqrt{25} = 5\,\mathrm{V}\,.$$

$$\tan(\theta) = \frac{-4}{3} \approx -1.33\,, \qquad \Rightarrow \theta = \tan^{-1}(-1.33) \approx -53°\,.$$

The result $3\angle 0°\mathrm{V} + 4\angle -90°\mathrm{V} = 5\angle -53°\mathrm{V}$ is drawn below



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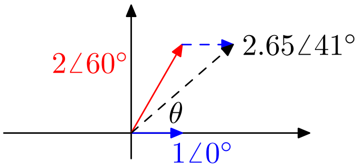
Exercise 3(c)

The sum of two currents $1\angle 0°\text{A}$ and $2\angle 60°\ \text{A}$ is a sum of two vectors $\mathbf{A}_1 = (1, 0)$ and $\mathbf{A}_2 = (2\cos(60), 2\sin(60)) = (1, \sqrt{3})$. Evaluating the sum $\mathbf{A}_R = \mathbf{A}_1 + \mathbf{A}_2 = (1+1, 0+\sqrt{3}) = (2, \sqrt{3})$ we find the length and the relative angle of the resulting current as

$$A_R = \sqrt{4+3} = \sqrt{7} \approx 2.65\text{A}\,.$$

$$\tan(\theta) = \frac{\sqrt{3}}{2} \approx 0.87\,, \qquad \Rightarrow \theta = \tan^{-1}(0.87) \approx 41°\,.$$

The result $1\angle 0°\text{A} + \angle 60°\text{A} = 2.65\angle 41°\text{A}$ is drawn below



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Exercise 3(d)

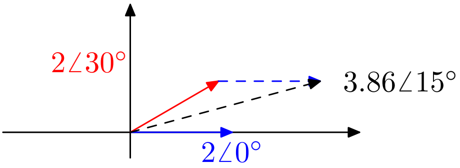
Since the amplitudes of the voltages $2\angle 0^\circ$ and $2\angle 30^\circ$ are the same, the sum must point halfway between them (here $\angle 15^\circ$). The amplitude depends upon the size of the angle between the phasors we are adding. The sum of the vectors $\mathbf{V}_1 = (2, 0)$ and

$$\mathbf{V}_2 = (2\cos(30), 2\sin(30)) = (\sqrt{3}, 1)$$

is $\mathbf{V}_1 + \mathbf{V}_2 = (2 + \sqrt{3}, 1)$ and the resulting amplitude is

$$V_R = \sqrt{(2+\sqrt{3})^2 + 1^2} \approx 3.86\,\mathrm{V}\,.$$

The result $2\angle 0^\circ\mathrm{V} + 2\angle 30^\circ\mathrm{V} = 3.86\angle 15^\circ\mathrm{V}$ is drawn below



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Solutions to Quizzes

Solution to Quiz:

If the RMS voltage is 400 kV the peak voltage can be found as follows. Since

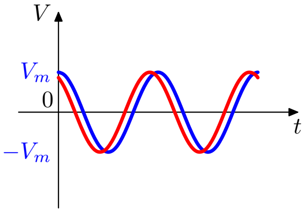
$$V_{\mathrm{RMS}} = \frac{V_m}{\sqrt{2}}\,,$$

therefore the peak voltage is

$$\begin{aligned} V_m &= \sqrt{2} \times V_{\mathrm{RMS}} \\ &= 1.41 \times 400\,\mathrm{kV} \\ &= 566\,\mathrm{kV}\,, \end{aligned}$$

End Quiz

Solution to Quiz:



In the diagram above the blue curve has a negative phase shift of 30° with respect to the red curve because the peaks or troughs of the red curve are measured at earlier times than their blue equivalents. Equivalent ways of stating this are:

- the blue curve lags the red curve by 30° ,
- the red curve leads the blue curve by 30° .

End Quiz

Solution to Quiz:

To add two voltages, reference $\mathbf{V}_1$ and $\mathbf{V}_2$, whose phasors are $3\angle 0^\circ\mathrm{V}$ and $4\angle 90^\circ$ V we find first the relative angle

$$\tan(\theta) = \frac{V_2}{V_1} = \frac{4}{3} \approx 1.33\,, \qquad \Rightarrow \theta = \tan^{-1}(1.33) \approx 53^\circ\,.$$

The length of the resulting voltage, $\mathbf{V}_R$, is

$$\begin{aligned} V_R^2 &= V_1^2 + V_2^2 = 3^2 + 4^2 = 25\,V^2\,, \\ \therefore\ V_R &= \sqrt{25} = 5\,V\,. \end{aligned}$$

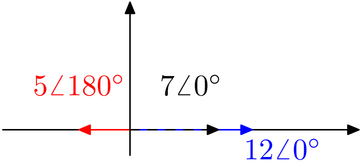
The result is thus:

$$3\angle 0^\circ\mathrm{V} + 4\angle 90^\circ\mathrm{V} = 5\angle 53^\circ\mathrm{V}\,.$$

End Quiz

Solution to Quiz:

Drawing the diagram with two voltages $12\angle 0^\circ$ and $5\angle 180^\circ$ V



we see that the resulting sum reads

$$12\angle 0^\circ + 5\angle 180^\circ = (12 - 5)\angle 0^\circ = 7\angle 0^\circ \text{ V}\,.$$

End Quiz