PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

Introduction to Forces

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The aim of this package is to provide a short self assessment programme for students who want to solve introductory problems about forces.

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Last Revision Date: August 15, 2005 Version 1.0

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

Newton's first law of motion states that an object will continue to stay at rest or move at constant speed in a straight line unless a force acts upon it.

A force has a size and a direction: it is therefore a **vector** quantity.

Newton's second law of motion states that a force $\boldsymbol{F}$ acting upon a body of mass m causes an acceleration $\boldsymbol{a}$ of the body given by:

$$\boldsymbol{F} = m\boldsymbol{a}$$

The acceleration is in the direction of the force.
The SI unit of force is the newton (N). One newton of force is defined as the force that causes a 1 kilogram mass to accelerate at $1\,\mathrm{ms}^{-2}$. (See the package on **Units**.)

Example 1 The tension $\boldsymbol{T}$ in a stretched rope is a force. It acts along the direction of the rope.
Notation: T is the (scalar) magnitude of the (vector) force $\boldsymbol{T}$.

2. Examples of Forces

Weight This is the force due to gravity. It is important to distinguish between mass and weight. If you were on the surface of the moon you would have a smaller weight, but your mass would be unchanged.

Your weight points straight down towards the center of the Earth. The size of your weight is given by $W = mg$ where g is the acceleration due to gravity. Its value is roughly $g = 10\,\mathrm{ms}^{-2}$.

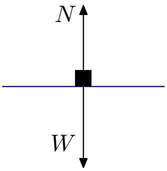
Example 2 The acceleration due to gravity on the surface of the moon is $g = 1.6\,\mathrm{ms}^{-2}$. What would be the weight of an astronaut with mass $m = 60\,\mathrm{kg}$?
Her weight would be $W = mg = 60 \times 1.6 = 96\,\mathrm{N}$. The direction of her weight would be towards the direction of the center of the moon.

Quiz On a new planet the same astronaut measures her weight to be $528\,\mathrm{N}$. What is the acceleration due to gravity on the planet?

(a) $8.8\,\mathrm{ms}^{-2}$, (b) $3.5\,\mathrm{ms}^{-2}$, (c) $588\,\mathrm{ms}^{-2}$, (d) $0.11\,\mathrm{ms}^{-2}$.

A book resting on a table stays at rest because of another force which balances the gravitational force. This force is the normal reaction.

Normal Reaction If a force, such as gravity, pushes an object against a surface, then the molecules in the surface will produce a force, called the normal reaction force, N, that stops the object penetrating the surface. The word normal refers to the force always being at right angles to the surface.



Example 3 If a book with mass 1.2 kg rests on a desk, what normal reaction force is produced by the desk?
The weight of the book is $W = mg = 1.2 \times 10 = 12$ N. The normal reaction which balances this is also 12 N and acts upwards at right angles to the desktop.

Quiz If you push down on a tabletop with a force of 30 N, what will be the magnitude of the normal reaction?

(a) 30 N , (b) 300 N , (c) 3 N , (d) 0.

Friction The force of friction resists attempts to move objects. If you push very gently at an object that is resting on a rough surface it will not move. The frictional force opposes the other force. If you push a little bit harder (exert a larger force) friction will increase to cancel the other force. If you push hard enough then, eventually, you may exert a force larger than the maximum possible friction and the object will start to accelerate.

Mathematically the above situation is expressed by an inequality:

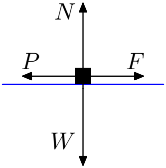
$$F \leq \mu N$$

where μ is called the coefficient of friction for that particular surface. The coefficient of friction is a dimensionless ratio (a force divided by a force) and is a number without any units.

Glossary:
a surface is *smooth* if frictional forces are small (negligible);
a surface is *rough* if frictional forces are significant (non-negligible).

Example 4 If a 15 kg mass is resting on a surface whose coefficient of friction is $\mu = 0.2$, find the frictional forces when the following forces are applied to the mass:

a) $P = 17\,\mathrm{N}$, b) $P = 29\,\mathrm{N}$, c) $P = 31\,\mathrm{N}$.



To answer these questions, we must first calculate the maximum possible friction. The normal reaction, N, is given by $N = mg = 15 \times 10 = 150\,\mathrm{N}$. Thus the maximum possible friction is

$$F_{\max} = \mu N = 0.2 \times 150 = 30\,\mathrm{N}\,.$$

a) The applied force $P = 17\,\mathrm{N}$ is less than the maximum friction. Thus the frictional force is $F = 17\,\mathrm{N}$ and the mass will not move.

b) The applied force $P = 28\,\mathrm{N}$ is still less than the maximum friction. Thus a frictional force of $F = 28\,\mathrm{N}$ will be produced and the mass will not move.

c) The applied force $P = 31\,\text{N}$ is more than the maximum friction. Thus the maximal frictional force of $F = 30\,\text{N}$ will be produced, but there will be a net force of $1\,\text{N}$ and the mass will start to accelerate in the direction of $\boldsymbol{P}$.

Exercise 1. A mass of $100\,\text{kg}$ rests on a rough surface with coefficient of friction $\mu = 0.4$. Find the frictional force when each of the following forces is applied to the mass. In each case state what will happen to the mass.

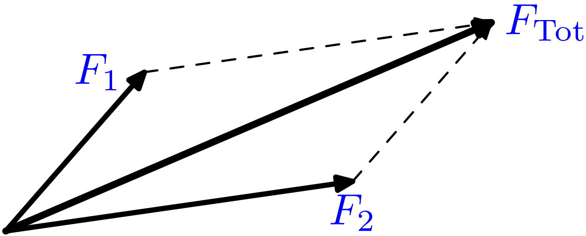
(a) $P = 10\,\text{N}$, (b) $P = 30\,\text{N}$, (c) $P = 40\,\text{N}$ (d) $P = 60\,\text{N}$

Quiz If the coefficient of friction of a level surface is $\mu = 0.25$ and a mass of $16\,\text{kg}$ is placed on it, how large a horizontal force can be applied to the mass before it starts to move?

(a) $640\,\text{N}$, (b) $2.5\,\text{N}$, (c) $160\,\text{N}$, (d) $40\,\text{N}$.

3. Forces as Vectors

If two forces act together on an object, their effect may be described as the action of one force. This is shown in the parallelogram of forces below (see the package **Introduction to Vectors**):

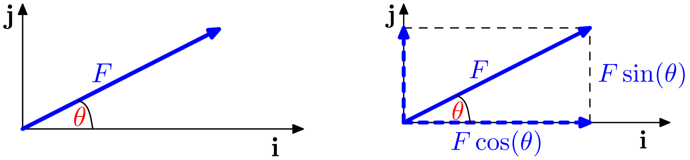


We write

$$\boldsymbol{F}_{\text{Tot}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 \,.$$

We call $\boldsymbol{F}_{\text{Tot}}$ the sum or **resultant** of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.

Consider the diagrams below



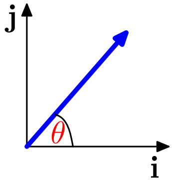
The force on the right has magnitude F and at an angle θ above the $\boldsymbol{i}$ direction. It can be understood as the sum of the two individual forces in the perpendicular x and y directions. The forces in those directions are called the **components** of the force.

$$\begin{aligned}\boldsymbol{F} &= F\cos(\theta)\boldsymbol{i} + F\sin(\theta)\boldsymbol{j} \\ &= F_x\boldsymbol{i} + F_y\boldsymbol{j}\end{aligned}$$

The magnitude F and the components F_x and F_y are related by Pythagoras' theorem:

$$F^2 = F_x^2 + F_y^2\,.$$

Example 5 Consider the force in the diagram below:



If the magnitude of the force is $F = 100\,\mathrm{N}$ and the angle $\theta = 60^\circ$, then the components of the force are:

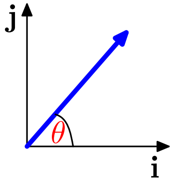
$$F_x = F\cos(\theta) \qquad \text{in the } \boldsymbol{i} \text{ direction}$$
$$F_y = F\sin(\theta) \qquad \text{in the } \boldsymbol{j} \text{ direction}$$

Substituting the numbers gives:

$$F_x = 100\cos(60) = 50\,N$$
$$F_y = 100\sin(60) = 87\,N$$

It may be checked that, as expected, $F_x^2 + F_y^2 = F^2 = 100^2$.

Exercise 2. Consider the diagram below:



Calculate the components F_x and F_y for the given values of the magnitude F and angle θ

(a) $F = 10\,\mathrm{N}, \quad \theta = 30^\circ$, (b) $F = 300\,\mathrm{N}, \quad \theta = 80^\circ$,
(c) $F = 3 \times 10^4\,\mathrm{N}, \quad \theta = 10^\circ$ (d) $F = 500\,\mathrm{N}, \quad \theta = 90^\circ$

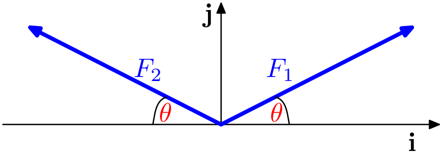
Quiz If a force may be written as

$$\boldsymbol{F} = 120\boldsymbol{i} + 50\boldsymbol{j}\ N$$

what is its magnitude F?

(a) 170 N, (b) 14,450 N, (c) 130 N, (d) 16,900 N.

It is important to be careful with the signs of the components. This is described on the next page.



The forces in the diagram above have the same magnitude but act in different directions, and we represent them as follows (see the package **Introduction to Vectors**):

$$\begin{aligned} \boldsymbol{F}_1 &= F_1\cos(\theta)\boldsymbol{i} + F_2\sin(\theta)\boldsymbol{i} \\ \boldsymbol{F}_2 &= -F_1\cos(\theta)\boldsymbol{i} + F_2\sin(\theta)\boldsymbol{i} \end{aligned}$$

Note the **negative sign** of the $\boldsymbol{i}$ component in $\boldsymbol{F}_2$.

The general rules are described on the next page.

Example 6 Consider the forces in the diagrams below:

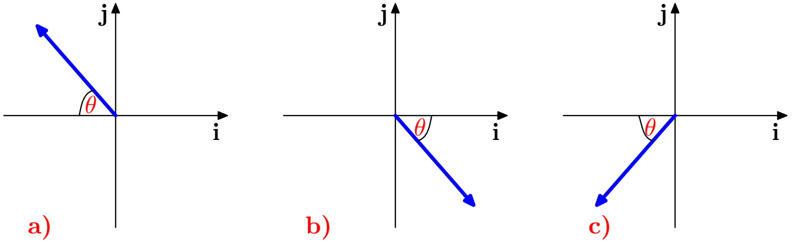


Diagram a) $\boldsymbol{F}$ may be written:

$$\boldsymbol{F} = -F\cos(\theta)\boldsymbol{i} + F\sin(\theta)\boldsymbol{j}\,.$$

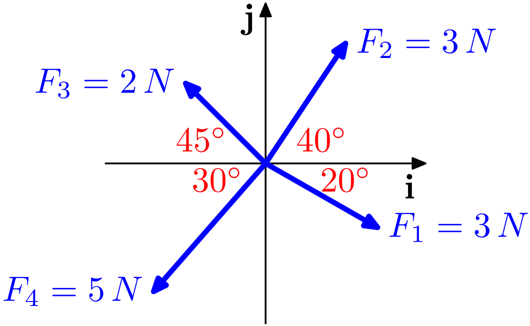
Diagram b) $\boldsymbol{F}$ may be written:

$$\boldsymbol{F} = F\cos(\theta)\boldsymbol{i} - F\sin(\theta)\boldsymbol{j}\,.$$

Diagram c) $\boldsymbol{F}$ may be written:

$$\boldsymbol{F} = -F\cos(\theta)\boldsymbol{i} - F\sin(\theta)\boldsymbol{j}\,.$$

Exercise 3. Write the forces in the diagram below in component form.

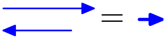


(a) F_1, (b) F_2, (c) F_3 (d) F_4

4. Combining Forces

Adding or subtracting two forces acting in the same direction (*collinear forces*) is simple: one adds or subtracts their magnitudes. This is how to add the forces in the diagrams below:

$$2\,N + 1.5\,N = 3.5\,N$$



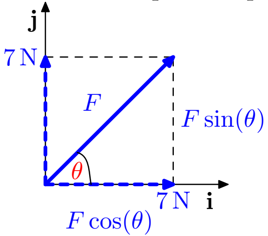
$$2\,N - 1.5\,N = 0.5\,N$$

To add two forces acting in different directions it is best to calculate their components and then add or subtract the components.

Example 7 To add the forces $\boldsymbol{F}_1 = 3\boldsymbol{i} + 2\boldsymbol{j}\,N$ and $\boldsymbol{F}_2 = 4\boldsymbol{i} + 5\boldsymbol{j}\,N$, we add the individual components:

$$\begin{aligned} \boldsymbol{F}_{\text{Tot}} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{i} + 5\boldsymbol{j} \\ &= (3+4)\boldsymbol{i} + (2+5)\boldsymbol{j} = 7\boldsymbol{i} + 7\boldsymbol{j}\,\text{N} \end{aligned}$$

One can reexpress this in terms of a magnitude and a direction as follows. Since $\boldsymbol{F}_{\rm Tot} = 7\boldsymbol{i} + 7\boldsymbol{j}\,{\rm N}$ both the $\boldsymbol{i}$ and $\boldsymbol{j}$ components are positive. This means that the force points up to the right:



The angle is given by

$$\tan(\theta) = \frac{7}{7} = 1\,, \quad \therefore \theta = \tan^{-1}(1) = 45^\circ\,.$$

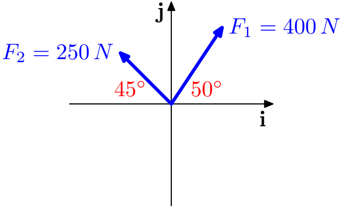
From Pythagoras' theorem, the magnitude is given by

$$F_{\rm Tot}^2 = 7^2 + 7^2 = 98\,,$$

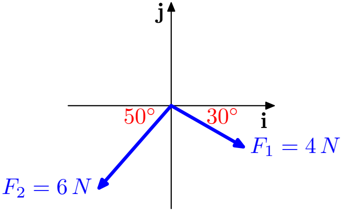
so that $F_{\rm Tot} = \sqrt{98} \approx 10\,{\rm N}$.

Exercise 4. Combine the following forces by decomposing the given forces into components and adding the components. Express your answer **both** in component form and **also** in terms of a magnitude and direction.

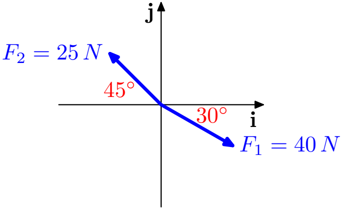
(a)



(b)



(c)



5. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. If it is found for a rough surface that a mass of 30 kg starts to move when a force of 60 N is applied to it, what is the coefficient of friction of the surface?

(a) 0.02 (b) $18,000$ (c) 0.2 (d) 0.05

2. What is the magnitude of the resultant of the two forces: $\boldsymbol{F}_1 = 30\boldsymbol{i} - 60\boldsymbol{j}$ N and $\boldsymbol{F}_2 = 10\boldsymbol{i} + 30\boldsymbol{j}$ N?

(a) 0 (b) 50 N (c) 76 N (d) 26 N

3. What angle does the resultant force of Q. 2 point in (measured anti-clockwise from the $\boldsymbol{i}$-axis)?

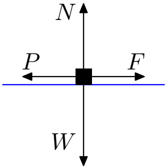
(a) 323° (b) 76° (c) 353° (d) 280°

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)

When the force $P = 10\,\text{N}$ is applied to a mass of $100\,\text{kg}$ resting on a rough surface with coefficient of friction $\mu = 0.4$ the normal reaction N is given by



$$N = W = mg = 100 \times 10 = 1000\,N\,,$$

and the maximum possible friction F_{max} is

$$F_{\text{max}} = \mu N = 0.4 \times 1000 = 400\,\text{N}\,.$$

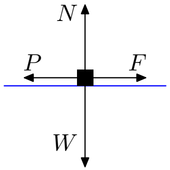
Since the applied force $P = 10\,\text{N}$ is less than the maximum friction $F_{\text{max}} = 400\,\text{N}$, the mass will not move.

Click on the **green** square to return

□

Exercise 1(b)

The applied force is now $P = 30\,\mathrm{N}$. The frictional force produced is, therefore, also $F = 30\,\mathrm{N}$, which is still less than the maximum possible friction $F_{\max}$



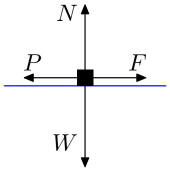
$$F_{\max} = \mu N = 0.4 \times 1000 = 400\,\mathrm{N}\,.$$

Thus the mass will not move.

Click on the **green** square to return

Exercise 1(c)

The force applied in this case is $P = 40\,\mathrm{N}$ and the frictional force produced will be $F = 40\,\mathrm{N}$. This force is less than the maximum possible friction $F_{\max}$



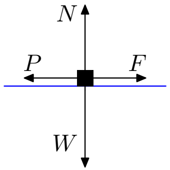
$$F_{\max} = \mu N = 0.4 \times 1000 = 400\,\mathrm{N}\,.$$

Thus the mass will still stay at rest.

Click on the **green** square to return □

Exercise 1(d)

In this case, the force applied to the mass is $P = 60\,\mathrm{N}$ and the resulting frictional force is $F = 60\,\mathrm{N}$. Since this is, again, less than the maximum possible friction $F_{\max} = 400\,\mathrm{N}$, the mass will stay at rest.



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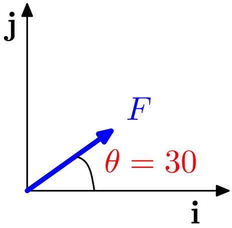
Exercise 2(a)



Consider the force with magnitude $F = 10\,\mathrm{N}$ at an angle of $\theta = 30^\circ$ to the $\boldsymbol{i}$-direction.

The components of the force are:

$$\begin{aligned} F_x &= F\cos(\theta) = 10\cos(30) = 10 \times \frac{\sqrt{3}}{2} \approx 8.66\,\mathrm{N} \\ F_y &= F\sin(\theta) = 10\sin(30) = 10 \times \frac{1}{2} = 5\,\mathrm{N} \end{aligned}$$

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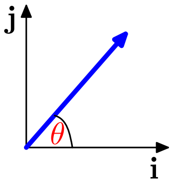
□

Exercise 2(b)



The force shown in the diagram has magnitude $F = 300\,\mathrm{N}$ and angle $\theta = 80^\circ$

The components of the force are:

$$F_x = F\cos(\theta) = 300\cos(80) \approx 300 \times 0.173 = 51.9\,\mathrm{N}$$
$$F_y = F\sin(\theta) = 300\sin(80) \approx 300 \times 0.984 = 295.2\,\mathrm{N}$$

Click on the **green** square to return

□

Exercise 2(c)

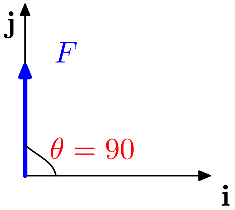
Consider the force with magnitude $F = 10^4\,\mathrm{N}$ and angle $\theta = 10^\circ$ with the $\boldsymbol{i}$-direction.
The components of the force are:

$$
\begin{aligned}
F_x &= F\cos(\theta) = 10^4\cos(10) \approx 10^4 \times 0.984807 = 9848.07\,\mathrm{N}\\
F_y &= F\sin(\theta) = 10^4\sin(10) \approx 10^4 \times = 0.173648 = 1736.48\,\mathrm{N}
\end{aligned}
$$

Click on the **green** square to return □

Exercise 2(d)

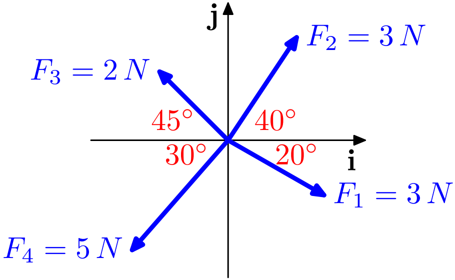
If the force has magnitude $F = 500\,\mathrm{N}$ and the angle θ with the $\boldsymbol{i}$ -direction is $\theta = 90^\circ$ then the components of the force are:



$$\begin{aligned} F_x &= F\cos(\theta) = 500\cos(90) = 500 \times 0 = 0 \\ F_y &= F\sin(\theta) = 500\sin(90) = 500\,\mathrm{N} \end{aligned}$$

Click on the **green** square to return □

Exercise 3(a)



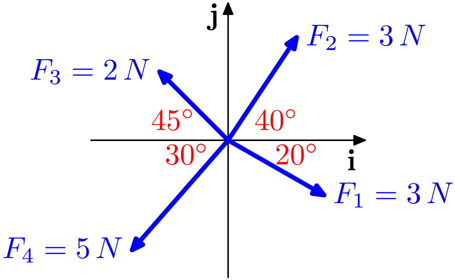
The force F_1 can be written in components as

$$F_1 = 3 \times \cos(20)\boldsymbol{i} - 3 \times \sin(20)\boldsymbol{j} \; = (2.82\boldsymbol{i} - 1.02\boldsymbol{j})\,\mathrm{N}\,.$$

Click on the **green** square to return

□

Exercise 3(b)



From the diagram above the force F_2 can be written in components as

$$F_2 = 3 \times \cos(40)\boldsymbol{i} + 3 \times \sin(40)\boldsymbol{j} \ = (2.29\boldsymbol{i} + 1.92\boldsymbol{j})\,\mathrm{N}\,.$$

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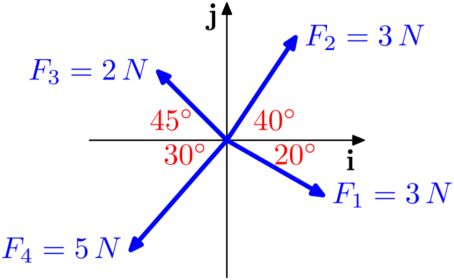
Exercise 3(c)



From the diagram above the force F_3 can be written in components as

$$F_3 = -2 \times \cos(45)\boldsymbol{i} + 2 \times \sin(45)\boldsymbol{j} \; = \left(-\sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j}\right) \text{ N}\,.$$

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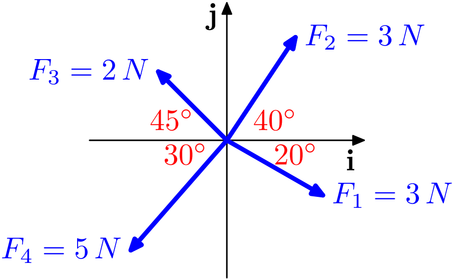
Exercise 3(d)



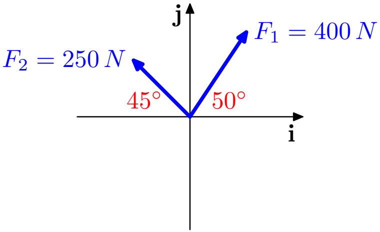
From the diagram above the force F_4 can be written in components as

$$F_4 = -5 \times \cos(30)\boldsymbol{i} - 5 \times \sin(30)\boldsymbol{j} = \left(-\frac{5\sqrt{3}}{2}\,\boldsymbol{i} - \frac{5}{2}\,\boldsymbol{j}\right) \text{N}\,.$$

Click on the **green** square to return □

Exercise 4(a)

The component forms of the forces in the picture are



$$\boldsymbol{F}_1 = 400\,(\cos(50)\boldsymbol{i} + \sin(50)\boldsymbol{j})\ ,$$

$$\boldsymbol{F}_2 = 250\,(-\cos(45)\boldsymbol{i} + \sin(45)\boldsymbol{j})\ .$$

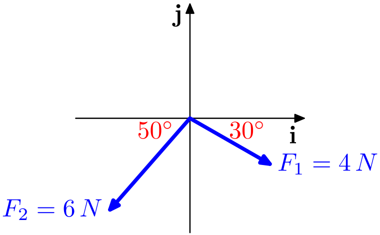
To add the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ we add the components

$$\begin{aligned}
\boldsymbol{F}_{12} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 \\
&= (400\cos(50) - 250\cos(45))\,\boldsymbol{i} + (400\sin(50) + 250\sin(45))\boldsymbol{j} \\
&= (257.1 - 176.8)\boldsymbol{i} + (306.4 + 176.8)\boldsymbol{j} \\
&= 80.3\boldsymbol{i} + 483.2\boldsymbol{j}\ \text{N}
\end{aligned}$$

Click on the **green** square to return

□

Exercise 4(b)



The component forms of the forces in the picture are

$$\boldsymbol{F}_1 = 4\,(\cos(30)\boldsymbol{i} - \sin(30)\boldsymbol{j})\ ,$$

$$\boldsymbol{F}_2 = 6\,(-\cos(50)\boldsymbol{i} - \sin(50)\boldsymbol{j})\ .$$

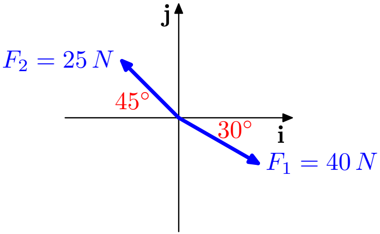
The total force is thus

$$\begin{aligned}
\boldsymbol{F}_{12} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 \\
&= (4\cos(30) - 6\cos(50))\,\boldsymbol{i} + (-4\sin(30) - 6\sin(50))\boldsymbol{j} \\
&= (3.46 - 3.86)\boldsymbol{i} + (-2 + 4.60)\boldsymbol{j} \\
&= -0.4\boldsymbol{i} + 2.6\boldsymbol{j}\ \mathrm{N}
\end{aligned}$$

Click on the **green** square to return

Exercise 4(c)

The component forms of the forces in the diagram are



$$\boldsymbol{F}_1 = 40\,(\cos(30)\boldsymbol{i} - \sin(30)\boldsymbol{j})\ ,$$

$$\boldsymbol{F}_2 = 25\,(-\cos(45)\boldsymbol{i} + \sin(45)\boldsymbol{j})\ .$$

To combine forces we add the components

$$\begin{aligned}
\boldsymbol{F}_{12} &= \boldsymbol{F}_1 + \boldsymbol{F}_2 \\
&= (40\cos(30) - 25\cos(45))\,\boldsymbol{i} + (-40\sin(30) + 25\sin(45))\boldsymbol{j} \\
&= (34.6 - 17.7)\boldsymbol{i} + (-20 + 17.7)\boldsymbol{j} \\
&= 16.9\boldsymbol{i} - 2.3\boldsymbol{j}\ \mathrm{N}
\end{aligned}$$

Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz:

If the measurement of the weight of a woman with mass $m = 60\,\text{kg}$ gives the result $528\,\text{N}$ then one can find the acceleration g on this planet from the basic equation

$$\mathbf{F} = m\,g\,.$$

Using this data we have

$$\begin{aligned} 528 &= 60\ \times g\,, \\ \therefore g &= \frac{528\ \text{N}}{60\text{kg}} = 8.8\text{ms}^{-2}\,. \end{aligned}$$

This is the acceleration on the planet Venus. End Quiz

Solution to Quiz:
Pushing the tabletop with force of W= 30 N in such a way that table does not move means that the tabletop resists with exactly the same normal reaction

$$W = N\,.$$

End Quiz

Solution to Quiz:

When a mass of $m = 16\,\mathrm{kg}$ is placed on a level surface with coefficient of friction $\mu = 0.25$ the normal reaction N is given by

$$N = mg = 16 \times 10 = 160\,N\,.$$

The maximum possible friction $F_{\max}$ is

$$F_{\max} = \mu N = 0.25 \times 160 = \frac{160}{4} = 40\,\mathrm{N}\,.$$

In order to move this mass a horizontal force, P, larger than $40\,\mathrm{N}$, should be applied. End Quiz

Solution to Quiz:

The force

$$\boldsymbol{F} = 120\boldsymbol{i} + 50\boldsymbol{j}\,\mathrm{N}$$

has the following components:

$$F_x = 120\mathrm{N} \qquad \text{and} \qquad F_y = 50\mathrm{N}\,.$$

The magnitude F is therefore

$$\begin{aligned} \mathrm{F} &= \sqrt{F_x^2 + F_y^2} = \sqrt{(120)^2 + (50)^2} \\ &= \sqrt{(120)^2 + (50)^2} \\ &= \sqrt{16900} = 130\,\mathrm{N}\,. \end{aligned}$$

End Quiz