Intermediate Mathematics

Series Tests

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The aim of this package is to provide a short self assessment programme for students who want to apply various tests to study the convergence properties of infinite series.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

If an infinite series $\sum_{r=1}^{\infty} a_r$ has a finite sum we say it converges (if not, it diverges). The sum of the first $n-$terms of the series is called the $n-$th partial sum. If the sequence of partial sums approaches a limit $\lim_{n\to\infty} s_n \to s$, then s is the sum of the series.

The sums of some familiar series are known, e.g., the geometric series

$$\sum_{r=0}^{\infty} x^r = \frac{1}{1-r}\,, \quad \text{if } |x| < 1\,.$$

Mathematicians have developed many tests to see whether series have finite sums or not. This package reviews some of these tests.

First, here is an example of one of the many limits that we will need to study the tests.

Example 1: As $r \to \infty$, what is the limit of $\dfrac{r^2+1}{r^2+3}$? Both the numerator and denominator blow up as r gets larger. Naively this gives:

$$\frac{r^2+1}{r^2+3} \to \frac{\infty}{\infty}\,,$$

which is meaningless! Instead divide top and bottom by r^2:

$$\frac{r^2+1}{r^2+3} = \frac{\dfrac{r^2}{r^2}+\dfrac{1}{r^2}}{\dfrac{r^2}{r^2}+\dfrac{3}{r^2}} = \frac{1+\dfrac{1}{r^2}}{1+\dfrac{3}{r^2}}$$

Now it is safe to take the limit $r \to \infty$ since

$$\frac{1+\dfrac{1}{r^2}}{1+\dfrac{3}{r^2}} \to \frac{1+0}{1+0} = 1\,,$$

and this limit is well-defined.

2. Non-Null Test

The first test we introduce states that:

> if $\sum_{r=1}^{\infty} a_r$ is convergent then $\{a_n\}$ is a null sequence.

It is important to realise that this test only states that if as $n \to \infty$, $a_n \not\to 0$, the series will diverge. It does **not** say that the series must converge if $a_n \to 0$. (This is therefore a test for divergence!)

Example 2: The series

$$\sum_{r=1}^{\infty} \frac{r}{r+1}$$

will not converge since as $n \to \infty$

$$\frac{r}{r+1} = \frac{1}{1+\frac{1}{r}} \to \frac{1}{1+0} = 1\,.$$

Proof of the Non-Null Test: The n-th partial sum, s_n, and the $(n-1)$-th partial sum are:

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n \\ s_{n-1} &= a_1 + a_2 + \cdots + a_{n-1} \end{aligned}$$

Subtracting these two equations gives:

$$s_n - s_{n-1} = a_n \, .$$

If the series converges to a limit s, then as $n \to \infty$, both $s_n \to s$ and also $s_{n-1} \to s$. Hence

$$a_n = s_n - s_{n-1} \to s - s = 0 \, .$$

So for any convergent series, a_n must approach zero.

Quiz Use the **non-null test** to select a series which must diverge.

(a) $\displaystyle\sum_{j=1}^{\infty} \frac{6j}{j^3+3}$, (b) $\displaystyle\sum_{j=1}^{\infty} \frac{j^2-3}{3+j^2}$, (c) $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j^2+4}$, (d) $\displaystyle\sum_{j=1}^{\infty} \frac{j^2+3}{2+j^4}$.

Exercise 1. What, if anything, does the **non-null test** say about the following series? (click on the **green** letters for solutions):

(a) $\displaystyle\sum_{w=1}^{\infty}\frac{1}{w}$, (b) $\displaystyle\sum_{j=1}^{\infty}\frac{j(j+1)}{2j^2-1}$,

(c) $\displaystyle\sum_{s=1}^{\infty}\frac{10^s}{8^s}$, (d) $\displaystyle\sum_{r=0}^{\infty}\frac{2^r}{3^r}$,

(e) $\displaystyle\sum_{r=0}^{\infty}\frac{100^r}{r!}$, (f) $\displaystyle\sum_{j=1}^{\infty}\frac{j^j}{j^{200}}$.

Quiz Which of the following series does the **non-null test** give no information about?

(a) $\displaystyle\sum_{j=1}^{\infty}\frac{6^j}{j^6}$, (b) $\displaystyle\sum_{w=1}^{\infty}\frac{w+w^2}{w^2+10}$, (c) $\displaystyle\sum_{s=1}^{\infty}\frac{s^5}{s!}$, (d) $\displaystyle\sum_{r=1}^{\infty}\frac{2r^{10}}{r(r^9+r^8)}$.

3. Comparison and Limit Comparison Tests

Sometimes it is possible to compare a series to another series whose properties are already known. There are two tests of this sort which we will state in turn without proof.

The **comparison test** states

if $0 \le a_n \le b_n$, for all $n \in \mathbb{N}$ then:

$$\sum_{n=1}^{\infty} b_n \quad \text{convergent} \Rightarrow \quad \sum_{n=1}^{\infty} a_n \quad \text{convergent}.$$

$$\sum_{n=1}^{\infty} a_n \quad \text{divergent} \Rightarrow \quad \sum_{n=1}^{\infty} b_n \quad \text{divergent}.$$

Example 3: It can be shown that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent. Hence $\sum_{r=1}^{\infty} \frac{1}{r^p}$ must also converge if $p > 2$, since then $\frac{1}{r^p} < \frac{1}{r^2}$.

Example 4: Since the Harmonic Series $\sum_{r=1}^{\infty} \frac{1}{r}$ is divergent, $\sum_{r=1}^{\infty} \frac{1}{r^p}$ must also diverge if $p \leq 1$, since then $\frac{1}{r^p} > \frac{1}{r}$.

EXERCISE 2. Use the **comparison test** and the fact that

$$\sum_{r=1}^{\infty} \frac{1}{r^p} \qquad \begin{cases} p > 1 & \text{series converges} \\ p \leq 1 & \text{series diverges} \end{cases}$$

to determine the convergence or otherwise of the following series. (click on the **green** letters for solutions):

(a) $\sum_{w=1}^{\infty} \frac{1}{(w+2)^2}$,

(b) $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}$,

(c) $\sum_{s=1}^{\infty} \frac{1+\cos(s)}{s^3}$,

(d) $\sum_{r=0}^{\infty} \frac{1}{r(r+1)}$, The **limit comparison test** states that if $a_n > 0$ and $b_n > 0$ for all n and

$$\text{if } \frac{a_n}{b_n} \to L \neq 0 \text{ then}$$

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ convergent } \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ convergent}$$

In other words *either* both series are divergent *or* both are convergent.

Example 5: Compare $\sum_{r=1}^{\infty} \frac{r+2}{r^2+3}$ with the Harmonic Series, $\sum_{r=1}^{\infty} \frac{1}{r}$ which is a standard example of a divergent series. The ratio of the terms is:

$$\frac{\frac{r+2}{r^2+3}}{\frac{1}{r}} = \frac{r(r+2)}{r^2+3}\,.$$

The limit is

$$\lim_{r\to\infty} \frac{r^2+2}{r^2+3} = \frac{1+\dfrac{2}{r^2}}{1+\dfrac{3}{r^2}} \to 1\,.$$

This finite limit implies either both series diverge or both converge. Since the Harmonic Series $\sum_{r=1}^{\infty}\frac{1}{r}$ is known to diverge, then $\sum_{r=1}^{\infty}\frac{r+2}{r^2+3}$ must also diverge.

Exercise 3. Use the **limit comparison test** and the appropriate series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

to determine the convergence or otherwise of the following series. (click on the **green** letters for solutions):

(a) $\sum_{n=1}^{\infty}\frac{1}{2n^3-n^2}$,

(b) $\sum_{n=1}^{\infty}\frac{n^4+2}{2n^5+n^2}$,

4. Ratio Test

For a series of positive terms $\sum_{n=1}^{\infty} a_n$ the following is true:

$$\text{if} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L\,, \quad \text{then} \quad \begin{cases} \text{if } L > 1 & \text{the series diverges} \\ \text{if } L < 1 & \text{the series converges} \\ \text{if } L = 1 & \text{the test is inconclusive} \end{cases}$$

Example 6: In the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$, the ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{n} \times \frac{2^n}{2^{n+1}} = \frac{n+1}{2n} = \frac{1+\frac{1}{n}}{2}\,,$$

and this converges to $\frac{1}{2}$ as $n \to \infty$. The **ratio test** thus states that this series converges.

Exercise 4. What, if anything, does the **ratio test** say about the following series? (click on the **green** letters for solutions):

(a) $\displaystyle\sum_{w=1}^{\infty} \frac{1}{w}$, (b) $\displaystyle\sum_{w=1}^{\infty} \frac{1}{w^2}$,

(c) $\displaystyle\sum_{s=1}^{\infty} s\mathrm{e}^{-s}$, (d) $\displaystyle\sum_{r=0}^{\infty} x^r$, (assume $x > 0$)

(e) $\displaystyle\sum_{r=0}^{\infty} \frac{100^r}{r!}$, (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Hint: in (f) you may use that as $n \to \infty$, $(1 + \frac{1}{n})^n \to \mathrm{e}$, the base of natural logarithms.

Quiz Select a series below which, by the **ratio test**, converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^w}{w}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{100}}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{n+1}$.

5. Final Quiz

Begin Quiz In each of the following questions choose the series which can be shown to diverge using the given test.

1. Non-null test

(a) $\displaystyle\sum_{i=1}^{\infty}\frac{3}{i}$, (b) $\displaystyle\sum_{j=1}^{\infty}\frac{j+2}{30+j}$, (c) $\displaystyle\sum_{i=1}^{\infty}\mathrm{e}^{-i}$, (d) $\displaystyle\sum_{j=1}^{\infty}j^{-j}$.

2. Ratio test

(a) $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i}$, (b) $\displaystyle\sum_{j=1}^{\infty}j$, (c) $\displaystyle\sum_{r=1}^{\infty}\frac{r!}{(r+1)!}$, (d) $\displaystyle\sum_{w=1}^{\infty}\frac{\mathrm{e}^{w}}{w}$.

3. Ratio test

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$, (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{3n!-1}$, (d) $\displaystyle\sum_{n=1}^{\infty}n\mathrm{e}^{-2n}$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) In the series $\sum_{w=1}^{\infty} \frac{1}{w}$ the term a_w vanishes as $w \to \infty$:

$$a_w \to 0$$

Hence the **non-null test** tells us *nothing* about this series.

In fact this series, which is called the **Harmonic Series**, diverges! This is despite the individual terms tending to zero. They do *not* vanish quickly enough for the series to converge. This is an important series which is widely used.

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□

Exercise 1(b) In the series $\sum_{j=1}^{\infty} \frac{j(j+1)}{2j^2-1}$:

$$\begin{aligned} a_j &= \frac{j(j+1)}{2j^2-1} = \frac{j^2+j}{2j^2-1} \\ &= \frac{1+\frac{1}{j}}{2-\frac{1}{j}}\,, \end{aligned}$$

where in the last step we divided both top and bottom by j^2. Hence as $j \to \infty$:

$$\begin{aligned} a_j &= \frac{1+\frac{1}{j}}{2-\frac{1}{j}} \to \frac{1+0}{2-0} \\ \therefore\ a_j &\to \frac{1}{2}\,. \end{aligned}$$

Since this does not vanish, the **non-null test** states that this series **diverges**.

Click on the **green** square to return □

Exercise 1(c) In the series $\sum_{s=1}^{\infty} \frac{10^s}{8^s}$, the terms may be written as:

$$\frac{10^s}{8^s} = \left(\frac{10}{8}\right)^s = 1.25^s\,.$$

Now for $s \geq 1$, $(1.25)^s \geq 1.25$ so the **non-null test** states that this series **diverges**.

Click on the **green** square to return □

Exercise 1(d) In the series $\sum_{r=0}^{\infty} \frac{2^r}{3^r}$, the terms may be written as:

$$\frac{2^r}{3^r} = \left(\frac{2}{3}\right)^r .$$

Recall that if $|x| < 1$, then $x^r \to 0$ as $r \to \infty$. Hence

$$\lim_{r\to\infty} a_r \to 0 .$$

Since this vanishes, the **non-null test** tells us nothing.

This example is actually a geometric series and is convergent.

Click on the **green** square to return □

Exercise 1(e) In the series $\sum_{r=1}^{\infty} \frac{100^r}{r!}$:

$$a_r = \frac{100^r}{r!}\,.$$

Now if $r > 100$ the denominator involves a product of r numbers with some being greater than 100, while in the numerator we have a product of 100 multiplied r times. This shows that for r sufficiently large $r! > 100^r$, i.e.,

$$\lim_{r\to\infty} \frac{100^r}{r!} \to 0\,.$$

Thus $a_r \to 0$ and the **non-null test** tells us nothing about this series (we will treat it below with the ratio test).

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Exercise 1(f) In the series $\sum_{j=1}^{\infty} \frac{j^j}{j^{200}}$:

$$a_j = \frac{j^j}{j^{200}} .$$

This shows that $a_{200} = 1$, $a_{201} = 201$, $a_{202} = 202^2$ etc. It is clear that as j increases the terms a_j get larger and larger. Thus

$$\lim_{j \to \infty} \frac{j^j}{j^{200}} \to \infty .$$

The **non-null test** therefore tells us that this series diverges.

Click on the **green** square to return □

Exercise 2(a) To analyse the series $\sum_{w=1}^{\infty} \frac{1}{(w+2)^2}$, use the inequality

$$(w+2)^2 > w^2 \quad \text{if } w > 0\,.$$

This implies that

$$\frac{1}{(w+2)^2} < \frac{1}{w^2}$$

Therefore the terms a_w satisfy

$$a_w = \frac{1}{(w+2)^2} < \frac{1}{w^2}$$

so, by the **comparison test**, this series must be convergent since $\sum_{w=1}^{\infty} \frac{1}{w^2}$ is convergent.

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Exercise 2(b) To study the series $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}$ recall that

$$\sqrt{j} < j \quad \text{if } j > 1\,,$$

so that

$$\frac{1}{\sqrt{j}} > \frac{1}{j}\,.$$

Therefore the terms a_j satisfy

$$a_j = \frac{1}{\sqrt{j}} > \frac{1}{j}\,.$$

The **comparison test** implies that this series must be divergent since the Harmonic Series $\sum_{j=1}^{\infty} \frac{1}{j}$ is divergent.

Click on the **green** square to return □

Exercise 2(c) To see that the series $\sum_{s=1}^{\infty} \frac{1+\cos(s)}{s^3}$ is convergent, note that:

$$-1 \le \cos(s) \le 1 \quad \Rightarrow \quad 0 \le 1+\cos(s) \le 2\,.$$

This implies that

$$0 \le \frac{1+\cos(s)}{s^3} \le \frac{2}{s^3}\,.$$

Since $\sum_{s=1}^{\infty} \frac{1}{s^3}$ is convergent, the comparison test says that $\sum_{s=1}^{\infty} \frac{1+\cos(s)}{s^3}$ **converges** too.

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Exercise 2(d) To see that the series $\sum_{r=0}^{\infty} \frac{1}{r(r+1)}$ is convergent, note that

$$r(r+1) = r^2 + r \geq r^2 \qquad (\text{since } r \geq 0)\,,$$

so that:

$$0 \leq \frac{1}{r(r+1)} \leq \frac{1}{r^2}\,.$$

Since $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent, the **comparison test** says that $\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$ **converges** too.

Click on the **green** square to return □

Exercise 3(a) In the series $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n^2}$ the ratio of the term a_n with the corresponding term b_n in $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is:

$$\frac{\frac{1}{2n^3 - n^2}}{\frac{1}{n^3}} = \frac{n^3}{2n^3 - n^2}$$

This is equal to

$$\frac{n^3}{2n^3 - n^2} = \frac{1}{2 - \frac{1}{n}}$$

The limit of this is $\frac{1}{2}$. So from the **limit comparison test**, since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, $\sum_{n=1}^{\infty} \frac{1}{2n^3 - n^2}$ is also **convergent**.

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Exercise 3(b) In the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+2}{2n^5+n^2}$ the ratio of the term a_n with the corresponding term in $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is:

$$\frac{\dfrac{n^4+2}{2n^5+n^2}}{\dfrac{1}{n}} = \frac{n^5+2n}{2n^5+n^2}$$

Dividing top and bottom by n^5 gives:

$$\frac{n^5+2n}{2n^5+n^2} = \frac{1+\dfrac{2}{n^4}}{2+\frac{1}{n^3}}$$

The limit of this is $\frac{1}{2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+2}{2n^5+n^2}$ diverges too. □

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Exercise 4(a) In the series $\sum_{w=1}^{\infty} \frac{1}{w}$ the ratio of the terms a_w and a_{w+1} is:

$$\frac{a_{w+1}}{a_w} = \frac{\frac{1}{w+1}}{\frac{1}{w}} = \frac{w}{w+1}\,.$$

Dividing top and bottom by w yields:

$$\frac{a_{w+1}}{a_w} = \frac{1}{1+\frac{1}{w}} \to \frac{1}{1+0} = 1\,.$$

Hence the **ratio test** tells us *nothing* about this example. (In fact it is the divergent Harmonic Series.)

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Exercise 4(b) In the series $\sum_{w=1}^{\infty} \frac{1}{w^2}$ the ratio of the terms a_w and a_{w+1} is:

$$\frac{a_{w+1}}{a_w} = \frac{\frac{1}{(w+1)^2}}{\frac{1}{w^2}} = \frac{w^2}{(w+1)^2}\,.$$

Expanding the denominator and dividing top and bottom by w^2 yields:

$$\frac{a_{w+1}}{a_w} = \frac{w^2}{w^2+2w+1} = \frac{1}{1+\frac{2}{w}+\frac{1}{w^2}}\,.$$

In the limit as $w \to \infty$ this becomes

$$\frac{a_{w+1}}{a_w} \to \frac{1}{1+0+0} = 1\,.$$

Thus the **ratio test** tells us nothing about this example. (It is actually convergent and the sum is $\pi^2/6$).

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Exercise 4(c) In the series $\sum_{s=1}^{\infty} s\mathrm{e}^{-s}$ the ratio of the terms a_s and a_{s+1} is:

$$\frac{a_{s+1}}{a_s} = \frac{(s+1)\mathrm{e}^{-(s+1)}}{s\mathrm{e}^{-s}} = \frac{s+1}{s}\mathrm{e}^{-s-1-(-s)} = \frac{s+1}{s}\mathrm{e}^{-1} .$$

Dividing top and bottom by s gives:

$$\lim_{s\to\infty} \frac{a_{s+1}}{a_s} \to \frac{1+0}{1}\mathrm{e}^{-1} = \frac{1}{\mathrm{e}} .$$

Since $\mathrm{e} > 1$, the limit of the ratio is < 1 and, by the **ratio test**, the series converges.

Click on the **green** square to return □

Exercise 4(d) In the geometric series $\sum_{r=0}^{\infty} x^r$ the ratio of the terms a_r and a_{r+1} is:

$$\frac{a_{r+1}}{a_r} = \frac{x^{r+1}}{x^r} = x\,,$$

which is independent of r. From the **ratio test**, we conclude that the geometric series will converge for any positive $x < 1$.

Click on the **green** square to return □

Exercise 4(e) In the series $\displaystyle\sum_{r=1}^{\infty} \frac{100^r}{r!}$:

$$a_r = \frac{100^r}{r!} \qquad \text{and} \qquad a_{r+1} = \frac{100^{r+1}}{(r+1)!}\,,$$

so that

$$\frac{a_{r+1}}{a_r} = \frac{\dfrac{100^{r+1}}{(r+1)!}}{\dfrac{100^r}{r!}} = \frac{r!}{(r+1)!} \times \frac{100^{r+1}}{100^r} = \frac{100}{r+1}\,.$$

Thus the ratio tends to zero and the **ratio test** tells us that this **series converges.**

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Exercise 4(f) In the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$:

$$a_n = \frac{n^n}{n!} \qquad \text{and} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!},$$

so that

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \frac{(n+1)(n+1)^n}{n^n} \times \frac{n!}{(n+1)n!},$$

using $(n+1)! = (n+1)n!$ and $(n+1)^{n+1} = (n+1)(n+1)^n$. This simplifies to

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n .$$

The limit as $n \to \infty$ is a definition of $\mathrm{e} \approx 2.71828$, the base of natural logarithms. Thus by the **ratio test** the series **diverges**.

Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: From the **non-null test**, the series $\displaystyle\sum_{j=1}^{\infty}\frac{j^2-3}{3+j^2}$ diverges. To see this consider

$$a_j = \frac{j^2-3}{3+j^2} = \frac{1-\frac{3}{j^2}}{\frac{3}{j^2}+1}\,,$$

where we divided top and bottom by j^2. For large j this becomes

$$a_j = \frac{1-\frac{3}{j^2}}{\frac{3}{j^2}+1} \to \frac{1-0}{0+1} = 1.$$

Since this does not vanish, the series must diverge.

For all the other series $a_j \to 0$ and the **non-null test** says *nothing* about the convergence or otherwise of these series. End Quiz

Solution to Quiz: If the **non-null test** gives us no information about a series, then the terms must tend to zero. This is true of **(c)**

$$\sum_{s=1}^{\infty} \frac{s^5}{s!}$$

where the terms

$$\frac{s^5}{s!} \to 0\,,$$

as $s \to \infty$.

It may be checked that none of the other terms tend to zero (the limits being, respectively, ∞, 1 and 2). Hence these series must all diverge. End Quiz

Solution to Quiz: The **ratio test** can be used to show that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. With $a_n = \frac{n!}{n^n}$ one has

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n^n}{n!}} = \frac{n^n}{(n+1)(n+1)^n} \times \frac{(n+1)n!}{n!}\,,$$

so that

$$\frac{a_{n+1}}{a_n} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n}\,,$$

which – see **Exercise 4f** – implies:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \to \frac{1}{\mathrm{e}} < 1\,.$$

Thus this series **converges** by the **ratio test**. (Note that although **(c)** is a convergent series, the ratio test cannot show it (one needs to use a comparison test). The other two series diverge.) End Quiz