

The Laplacian

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The aim of this package is to provide a short self assessment programme for students who want to apply the Laplacian operator.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page [Mathematics Support Materials](#).

1. Introduction (Grad, Div, Curl)

The **vector differential operator** ∇ , called “del” or “nabla”, is defined in three dimensions to be:

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$$

The result of applying this vector operator to a scalar field is called the **gradient of the scalar field**:

$$\text{grad}f(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

(See the package on **Gradients and Directional Derivatives**.)

The *scalar product* of this vector operator with a vector field $\mathbf{F}(x, y, z)$ is called the **divergence of the vector field**:

$$\text{div}\mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The *vector product* of the vector ∇ with a vector field $\mathbf{F}(x, y, z)$ is the **curl of the vector field**. It is written as $\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}$

$$\begin{aligned}\nabla \times \mathbf{F}(x, y, z) &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix},\end{aligned}$$

where the last line is a formal representation of the line above. (See also the package on **Divergence and Curl**.)

Here are some revision exercises.

EXERCISE 1. For $f = x^2y - z$ and $\mathbf{F} = xi - xyj + z^2k$ calculate the following (click on the **green** letters for the solutions).

(a) ∇f

(b) $\nabla \cdot \mathbf{F}$

(c) $\nabla \times \mathbf{F}$

(d) $\nabla f - \nabla \times \mathbf{F}$

2. The Laplacian

The **Laplacian** operator is defined as:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The **Laplacian** is a **scalar operator**. If it is applied to a **scalar field**, it generates a **scalar field**.

Example 1 The **Laplacian** of the scalar field $f(x, y, z) = xy^2 + z^3$ is:

$$\begin{aligned}\nabla^2 f(x, y, z) &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{\partial^2}{\partial x^2}(xy^2 + z^3) + \frac{\partial^2}{\partial y^2}(xy^2 + z^3) + \frac{\partial^2}{\partial z^2}(xy^2 + z^3) \\ &= \frac{\partial}{\partial x}(y^2 + 0) + \frac{\partial}{\partial y}(2xy + 0) + \frac{\partial}{\partial z}(0 + 3z^2) \\ &= 0 + 2x + 6z = 2x + 6z\end{aligned}$$

EXERCISE 2. Calculate the **Laplacian** of the following scalar fields: (click on the **green** letters for the solutions).

(a) $f(x, y, z) = 3x^3y^2z^3$ (b) $f(x, y, z) = \sqrt{xz} + y$

(c) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ (d) $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

Quiz Choose the **Laplacian** of $f(r) = \frac{1}{r^n}$ where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) $-\frac{1}{r^{n+2}}$

(b) $\frac{n}{r^{n+2}}$

(c) $\frac{n(n-1)}{r^{n+2}}$

(d) $\frac{n(n+5)}{r^{n+2}}$

The equation $\nabla^2 f = 0$ is called *Laplace's equation*. This is an important equation in science. From the above exercises and quiz we see that $f = \frac{1}{r}$ is a solution of Laplace's equation except at $r = 0$.

The Laplacian of a scalar field can also be written as follows:

$$\nabla^2 f = \nabla \cdot \nabla f$$

i.e., as the **divergence of the gradient of f** . To see this consider

$$\begin{aligned}\nabla \cdot \nabla f &= \nabla \cdot \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f\end{aligned}$$

EXERCISE 3. Find the **Laplacian** of the scalar fields f whose gradients ∇f are given below (click on the **green** letters for the solutions).

(a) $\nabla f = 2xz\mathbf{i} + x^2\mathbf{k}$

(b) $\nabla f = \frac{1}{yz}\mathbf{i} - \frac{x}{y^2z}\mathbf{j} - \frac{x}{yz^2}\mathbf{k}$

(c) $\nabla f = e^z\mathbf{i} + y\mathbf{j} + xe^z\mathbf{k}$

(d) $\nabla f = \frac{1}{x}\mathbf{i} + \frac{1}{y}\mathbf{j} + \frac{1}{z}\mathbf{k}$

3. The Laplacian of a Product of Fields

If a field may be written as a product of two functions, then:

$$\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u) \cdot (\nabla v)$$

A proof of this is given at the end of this section.

Example 2 The **Laplacian** of $f(x, y, z) = (x + y + z)(x - 2z)$ may be directly calculated from the above rule

$$\begin{aligned}\nabla^2 f(x, y, z) &= (\nabla^2(x + y + z))(x - 2z) + (x + y + z)\nabla^2(x - 2z) \\ &\quad + 2\nabla(x + y + z) \cdot \nabla(x - 2z)\end{aligned}$$

Now $\nabla^2(x + y + z) = 0$ and $\nabla^2(x - 2z) = 0$ so the first line on the right hand side vanishes.

To calculate the second line we note that

$$\nabla(x+y+z) = \frac{\partial(x+y+z)}{\partial x} \mathbf{i} + \frac{\partial(x+y+z)}{\partial y} \mathbf{j} + \frac{\partial(x+y+z)}{\partial z} \mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

and

$$\nabla(x - 2z) = \frac{\partial(x - 2z)}{\partial x} \mathbf{i} + \frac{\partial(x - 2z)}{\partial y} \mathbf{j} + \frac{\partial(x - 2z)}{\partial z} \mathbf{k} = \mathbf{i} - 2\mathbf{k}$$

and taking their scalar product we obtain

$$\begin{aligned} \nabla^2 f(x, y, z) &= 0 + 2\nabla(x + y + z) \cdot \nabla(x - 2z) \\ &= 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{k}) = 2(1 + 0 - 2) \\ &= -2. \end{aligned}$$

This example may be checked by expanding $(x + y + z)(x - 2z)$ and directly calculating the Laplacian.

EXERCISE 4. Use this rule to calculate the **Laplacian** of the scalar fields given below (click on the **green** letters for the solutions).

(a) $(2x - 5y + z)(x - 3y + z)$ (b) $(x^2 - y)(x + z)$

(c) $(y - z)(x^2 + y^2 + z^2)$ (d) $x\sqrt{x^2 + y^2 + z^2}$

Proof that $\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u) \cdot (\nabla v)$.

By definition $\nabla^2(uv) = \frac{\partial^2}{\partial x^2}(uv) + \frac{\partial^2}{\partial y^2}(uv) + \frac{\partial^2}{\partial z^2}(uv)$. Consider therefore:

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(uv) &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x}(uv) \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} \right) \\ &= \frac{\partial^2 u}{\partial x^2}v + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} \\ &= \frac{\partial^2 u}{\partial x^2}v + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} \end{aligned}$$

where the **product rule** was repeatedly used.

By symmetry we also have: $\frac{\partial^2}{\partial y^2}(uv) = \frac{\partial^2 u}{\partial y^2}v + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2}$ and

$\frac{\partial^2}{\partial z^2}(uv) = \frac{\partial^2 u}{\partial z^2}v + 2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2}$. Adding these results yields the desired result.

4. The Laplacian and Vector Fields

If the scalar Laplacian operator is applied to a **vector field**, it acts on each component in turn and generates a **vector field**.

Example 3 The **Laplacian** of $\mathbf{F}(x, y, z) = 3z^2\mathbf{i} + xyz\mathbf{j} + x^2z^2\mathbf{k}$ is:

$$\nabla^2 \mathbf{F}(x, y, z) = \nabla^2(3z^2)\mathbf{i} + \nabla^2(xyz)\mathbf{j} + \nabla^2(x^2z^2)\mathbf{k}$$

Calculating the components in turn we find:

$$\nabla^2(3z^2) = \frac{\partial^2}{\partial x^2}(3z^2) + \frac{\partial^2}{\partial y^2}(3z^2) + \frac{\partial^2}{\partial z^2}(3z^2) = 0 + 0 + 6 = 6$$

$$\nabla^2(xyz) = \frac{\partial^2}{\partial x^2}(xyz) + \frac{\partial^2}{\partial y^2}(xyz) + \frac{\partial^2}{\partial z^2}(xyz) = 0 + 0 + 0 = 0$$

$$\nabla^2(x^2z^2) = \frac{\partial^2}{\partial x^2}(x^2z^2) + \frac{\partial^2}{\partial y^2}(x^2z^2) + \frac{\partial^2}{\partial z^2}(x^2z^2) = 2z^2 + 0 + 2x^2$$

So the Laplacian of \mathbf{F} is:

$$\nabla^2 \mathbf{F} = 6\mathbf{i} + 0\mathbf{j} + (2z^2 + 2x^2)\mathbf{k} = 6\mathbf{i} + 2(x^2 + z^2)\mathbf{k}$$

Quiz Select from the answers below the Laplacian of the vector field $\mathbf{F} = x^3y\mathbf{i} + \ln(z)\mathbf{j} + \ln(xy)\mathbf{k}$.

- (a) $6xy\mathbf{i}$ (b) $6xy\mathbf{i} - \frac{1}{z^2}\mathbf{j} - \frac{x^2 + y^2}{x^2y^2}\mathbf{k}$
- (c) $3x\mathbf{i} + \frac{1}{z^2}\mathbf{j} - \left(\frac{1}{x^2} - \frac{1}{y^2}\right)\mathbf{k}$ (d) $y^3\mathbf{i} + x^2\mathbf{j} - \frac{y^2 - x^2}{z^2}\mathbf{k}$

Quiz Choose the Laplacian of $\mathbf{F} = 3x^2z\mathbf{i} - \sin(\pi y)\mathbf{j} + \ln(2x^3)\mathbf{k}$ at the point $(1, -2, 1)$?

- (a) 0 (b) $6\mathbf{i} - 3\mathbf{k}$ (c) $3\mathbf{i} - 3\mathbf{k}$ (d) $6\mathbf{i} - \pi^2\mathbf{j} - 6\mathbf{k}$

Quiz Which of the following choices is the Laplacian of the vector field $\mathbf{F} = \ln(y)\mathbf{i} + z^2\mathbf{j} - \sin(2\pi x)\mathbf{k}$ at $(1, 1, \pi)$?

- (a) $-\mathbf{i} + 2\mathbf{j}$ (b) 0 (c) $4\mathbf{j} + 4\pi^2\mathbf{k}$ (d) $-\mathbf{i} + \mathbf{j} + \mathbf{k}$

5. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. Choose the Laplacian of $f(\mathbf{r}) = 5x^3y^4z^2$.

(a) $30xy^4z^2 + 60x^3y^2z^2 + 10x^3y^4$ (b) $30x + 20y^2 + 10$

(c) $30xy^4z^2 + 75x^3y^2z^2 + 15x^3y^4$ (d) $30xy^4z^2 + 12x^3y^2z^2 + 15x^3y^4$

2. Choose the Laplacian of $f(x, y, z) = \ln(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) 0 (b) $\frac{2}{r^2}$ (c) $\frac{1}{2r^2}$ (d) $\frac{1}{r^2}$

3. Select the Laplacian of $\mathbf{F} = x^3\mathbf{i} + 7y\mathbf{j} - 3\sin(2y)\mathbf{k}$.

(a) $6x\mathbf{i}$ (b) $6x\mathbf{i} + 12\sin(2y)\mathbf{k}$

(c) $6x\mathbf{i} + 3\sin(2y)\mathbf{k}$ (d) $6x\mathbf{i} - 12\sin(2y)\mathbf{k}$

End Quiz

Solutions to Exercises

Exercise 1(a)

To find the **gradient** of the scalar field $f = x^2y - z$, we need the partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2y - z) = 2x^{2-1} \times y = 2xy, \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2y - z) = x^2 \times y^{1-1} = x^2, \\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}(x^2y - z) = 0 - z^{1-1} = -1.\end{aligned}$$

Therefore the **gradient** of $f = x^2y - z$ is

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}.\end{aligned}$$

Click on the **green** square to return



Exercise 1(b)

To find the **divergence** of the vector field $\mathbf{F} = xi - xyj + z^2\mathbf{k}$, we recognise that its components are

$$F_1 = x, \quad F_2 = -xy, \quad F_3 = z^2,$$

So the divergence is the **scalar** expression

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(z^2) \\ &= x^{1-1} - x \times y^{1-1} + 2 \times z^{2-1} \\ &= 1 - x + 2z.\end{aligned}$$

Click on the **green** square to return



Exercise 1(c)

The *curl* of the vector field \mathbf{F} whose components are

$$F_1 = x, \quad F_2 = -xy, \quad F_3 = z^2,$$

is given by the **vector** expression:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k} \\ &= \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(-xy) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x) \right) \mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial y}(x) \right) \mathbf{k} \\ &= (0 + 0) \mathbf{i} - (0 - 0) \mathbf{j} + (-y - 0) \mathbf{k} \\ &= -y\mathbf{k}.\end{aligned}$$

Click on the **green** square to return



Exercise 1(d)

To subtract the **curl** of the vector $\mathbf{F} = x\mathbf{i} - xy\mathbf{j} + z^2\mathbf{k}$ from the gradient of the scalar field $f = x^2y - z$

$$\nabla f - \nabla \times \mathbf{F}.$$

we use the results of **Exercise 1a** and **Exercise 1c**, where it was found that

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \nabla \times \mathbf{F} = -y\mathbf{k}.$$

Therefore the difference of these two vectors is

$$\begin{aligned}\nabla f - \nabla \times \mathbf{F} &= 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k} - (-y)\mathbf{k} \\ &= 2xy\mathbf{i} + x^2\mathbf{j} - (1 - y)\mathbf{k}.\end{aligned}$$

Click on the **green** square to return



Exercise 2(a)

The **Laplacian** of the scalar field $f = 3x^3y^2z^3$ is:

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{\partial^2}{\partial x^2}(3x^3y^2z^3) + \frac{\partial^2}{\partial y^2}(3x^3y^2z^3) + \frac{\partial^2}{\partial z^2}(3x^3y^2z^3) \\ &= \frac{\partial}{\partial x}(9x^2y^2z^3) + \frac{\partial}{\partial y}(6x^3yz^3) + \frac{\partial}{\partial z}(9x^3y^2z^2) \\ &= 18xy^2z^3 + 6x^3z^3 + 18x^3y^2z.\end{aligned}$$

Extracting common factors, the scalar $\nabla^2 f$ can also be written as

$$\begin{aligned}\nabla^2 f &= 6xz(3y^2z^2 + x^2z^2 + 3x^2y^2) \\ &= 6xz(3y^2(z^2 + x^2) + x^2z^2).\end{aligned}$$

Click on the **green** square to return



Exercise 2(b)

The **Laplacian** of the scalar field $f = \sqrt{xz} + y = x^{1/2}z^{1/2} + y$ is:

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2}{\partial x^2} \left(x^{\frac{1}{2}} z^{\frac{1}{2}} + y \right) + \frac{\partial^2}{\partial y^2} \left(x^{\frac{1}{2}} z^{\frac{1}{2}} + y \right) + \frac{\partial^2}{\partial z^2} \left(x^{\frac{1}{2}} z^{\frac{1}{2}} + y \right) \\ &= \frac{\partial}{\partial x} \left(\frac{1}{2} x^{(\frac{1}{2}-1)} z^{\frac{1}{2}} \right) + \frac{\partial}{\partial y} (1) + \frac{\partial}{\partial z} \left(\frac{1}{2} x^{\frac{1}{2}} z^{(\frac{1}{2}-1)} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{1}{2} x^{-\frac{1}{2}} z^{\frac{1}{2}} \right) + 0 + \frac{\partial}{\partial z} \left(\frac{1}{2} x^{\frac{1}{2}} z^{-\frac{1}{2}} \right) \\ &= \frac{1}{2} \left(-\frac{1}{2} \right) x^{(-\frac{1}{2}-1)} z^{\frac{1}{2}} + \frac{1}{2} \left(-\frac{1}{2} \right) x^{\frac{1}{2}} z^{(-\frac{1}{2}-1)} \\ &= -\frac{1}{4} x^{-\frac{3}{2}} z^{\frac{1}{2}} - \frac{1}{4} x^{\frac{1}{2}} z^{-\frac{3}{2}} .\end{aligned}$$

This result may be rewritten as

$$\nabla^2 f = -\frac{1}{4} x^{\frac{1}{2}} z^{\frac{1}{2}} (x^{-2} + z^{-2}) = -\frac{1}{4} \sqrt{xz} \left(\frac{1}{x^2} + \frac{1}{z^2} \right) .$$

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Exercise 2(c) To calculate $\nabla^2 \sqrt{x^2 + y^2 + z^2}$, define $u = x^2 + y^2 + z^2$, so $f = u^{1/2}$. From the chain rule we have

$$\frac{\partial f}{\partial x} = \frac{\partial u^{\frac{1}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = \frac{1}{2} u^{(\frac{1}{2}-1)} \times 2x = x u^{-\frac{1}{2}}.$$

Therefore the second derivative is (from the product and chain rules):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(x) \times u^{-\frac{1}{2}} + x \times \frac{\partial u^{-\frac{1}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = u^{-\frac{1}{2}} - x^2 u^{-\frac{3}{2}}.$$

Since f and u are symmetric in x, y and z we have

$$\frac{\partial^2 f}{\partial y^2} = u^{-\frac{1}{2}} - y^2 u^{-\frac{3}{2}} \quad \text{and} \quad \frac{\partial^2 f}{\partial z^2} = u^{-\frac{1}{2}} - z^2 u^{-\frac{3}{2}}.$$

Adding these results and using $x^2 + y^2 + z^2 = u$ yields

$$\begin{aligned} \nabla^2 f &= \left(u^{-\frac{1}{2}} - x^2 u^{-\frac{3}{2}}\right) + \left(u^{-\frac{1}{2}} - y^2 u^{-\frac{3}{2}}\right) + \left(u^{-\frac{1}{2}} - z^2 u^{-\frac{3}{2}}\right) \\ &= 3u^{-\frac{1}{2}} - (x^2 + y^2 + z^2)u^{-\frac{3}{2}} = 2u^{-\frac{1}{2}} = \frac{2}{\sqrt{x^2 + y^2 + z^2}}. \quad \square \end{aligned}$$

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Exercise 2(d) To find the **Laplacian** of $f = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, we again define $u = x^2 + y^2 + z^2$. From the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial u^{-\frac{1}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = -\frac{1}{2}u^{(-\frac{1}{2}-1)} \times 2x = -xu^{-\frac{3}{2}}.$$

Thus the second order derivative is:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial}{\partial x}(x) \times u^{-\frac{3}{2}} - x \times \frac{\partial u^{-\frac{3}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = -u^{-\frac{3}{2}} + 3x^2u^{-\frac{5}{2}}.$$

Due to the symmetry under interchange of x, y and z :

$$\frac{\partial^2 f}{\partial y^2} = -u^{-\frac{3}{2}} + 3y^2u^{-\frac{5}{2}}, \quad \frac{\partial^2 f}{\partial z^2} = -u^{-\frac{3}{2}} + 3z^2u^{-\frac{5}{2}}.$$

Therefore we find that the **Laplacian** of f vanishes:

$$\begin{aligned}\nabla^2 f &= \left(-u^{-\frac{3}{2}} + 3x^2u^{-\frac{5}{2}}\right) + \left(-u^{-\frac{3}{2}} + 3y^2u^{-\frac{5}{2}}\right) + \left(-u^{-\frac{3}{2}} + 3z^2u^{-\frac{5}{2}}\right) \\ &= -3u^{-\frac{3}{2}} + 3(x^2 + y^2 + z^2)u^{-\frac{5}{2}} = -3u^{-\frac{3}{2}} + 3u^{-\frac{3}{2}} = 0.\end{aligned}$$

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Exercise 3(a)

If the **gradient** of the scalar function f is $\nabla f = 2xz\mathbf{i} + x^2\mathbf{k}$, then the **Laplacian** of f is given by the **divergence** of this vector

$$\nabla^2 f = \operatorname{div}(\nabla f) .$$

Therefore the **Laplacian** of f is

$$\begin{aligned}\nabla^2 f &= \operatorname{div}(2xz\mathbf{i} + x^2\mathbf{k}) \\ &= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) \\ &= 2z + 0 + 0 \\ &= 2z .\end{aligned}$$

Click on the **green** square to return



Exercise 3(b)

If a scalar field f has gradient $\nabla f = \frac{1}{yz}\mathbf{i} - \frac{x}{y^2z}\mathbf{j} - \frac{x}{yz^2}\mathbf{k}$, its **Laplacian** is given by the **divergence**

$$\nabla^2 f = \operatorname{div}(\nabla f).$$

Using this formula we can find $\nabla^2 f$ as follows

$$\begin{aligned}\nabla^2 f &= \operatorname{div}\left(\frac{1}{yz}\mathbf{i} - \frac{x}{y^2z}\mathbf{j} - \frac{x}{yz^2}\mathbf{k}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{yz}\right) + \frac{\partial}{\partial y}\left(-\frac{x}{y^2z}\right) + \frac{\partial}{\partial z}\left(-\frac{x}{yz^2}\right) \\ &= (0) - (-2) \times \frac{x}{y^3z} - (-2) \times \frac{x}{yz^3} \\ &= 2x \frac{y^2 + z^2}{y^3z^3}.\end{aligned}$$

Click on the **green** square to return



Exercise 3(c)

As above if the gradient of f is

$$\nabla f = e^z \mathbf{i} + y \mathbf{j} + x e^z \mathbf{k}$$

its **Laplacian** is the **scalar divergence** of the vector ∇f :

$$\nabla^2 f = \operatorname{div}(\nabla f) .$$

Therefore

$$\begin{aligned} \nabla^2 f &= \operatorname{div}(e^z \mathbf{i} + y \mathbf{j} + x e^z \mathbf{k}) \\ &= \frac{\partial}{\partial x}(e^z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x e^z) \\ &= (0) + 1 + x e^z \\ &= 1 + x e^z . \end{aligned}$$

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Exercise 3(d)

If $\nabla f = \frac{1}{x}\mathbf{i} + \frac{1}{y}\mathbf{j} + \frac{1}{z}\mathbf{k}$, then the **Laplacian** of f is the **divergence** of the **gradient** of f :

$$\nabla^2 f = \operatorname{div}(\nabla f).$$

Therefore

$$\begin{aligned}\nabla^2 f &= \operatorname{div}\left(\frac{1}{x}\mathbf{i} + \frac{1}{y}\mathbf{j} + \frac{1}{z}\mathbf{k}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{z}\right) \\ &= -\frac{1}{x^2} - \frac{1}{y^2} - \frac{1}{z^2}.\end{aligned}$$

This may also be written as follows

$$\nabla^2 f = -\frac{x^2y^2 + z^2x^2 + y^2z^2}{x^2y^2z^2}.$$

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Exercise 4(a)

To find the **Laplacian** of $f = (2x - 5y + z)(x - 3y + z)$, we use

$$\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u) \cdot (\nabla v)$$

with $u = 2x - 5y + z$ and $v = x - 3y + z$. This implies

$$\nabla u = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$$

$$\nabla v = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

and so

$$\nabla^2 u = \nabla^2 v = 0$$

The above rule therefore gives:

$$\begin{aligned}\nabla^2 f = 2\nabla u \cdot \nabla v &= 0 + 0 + 2(2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \\ &= 2(2 + 15 + 1) \\ &= 36.\end{aligned}$$

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Exercise 4(b)

To find the **Laplacian** of $f = (x^2 - y)(x + z)$, we use

$$\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u) \cdot (\nabla v)$$

with $u = x^2 - y$ and $v = x + z$. This implies

$$\nabla u = 2xi - j$$

$$\nabla v = i + k$$

and so

$$\nabla^2 u = \nabla \cdot 2xi = \frac{\partial(2x)}{\partial x} = 2 \quad \text{and} \quad \nabla^2 v = 0$$

The above rule therefore gives:

$$\begin{aligned}\nabla^2 f &= 2(x + z) + 0 + 2(2i - j) \cdot (i + k) \\ &= 2(x + z) + 2(2) \\ &= 2(x + z + 2).\end{aligned}$$

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Exercise 4(c)

To find the **Laplacian** of $f = (y - z)(x^2 + y^2 + z^2)$, we use

$$\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u) \cdot (\nabla v)$$

with $u = y - z$ and $v = x^2 + y^2 + z^2$. This implies

$$\nabla u = \mathbf{j} - \mathbf{k} \quad \text{and} \quad \nabla^2 u = 0$$

while

$$\begin{aligned}\nabla v &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ \nabla^2 v &= 2 + 2 + 2 = 6\end{aligned}$$

The above rule therefore gives:

$$\begin{aligned}\nabla^2 f &= 0 + (y - z) \times 6 + 2(\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ &= 6 + 2(0 + 2 - 2) \\ &= 6.\end{aligned}$$

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Exercise 4(d) To find the **Laplacian** of $f = x\sqrt{x^2 + y^2 + z^2}$, we use

$$\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u) \cdot (\nabla v)$$

with $u = x$ and $v = \sqrt{x^2 + y^2 + z^2}$. Therefore $\nabla u = \mathbf{i}$ and $\nabla^2 u = 0$.

In Exercise 2(c) we saw $\nabla^2 v = -2/\sqrt{x^2 + y^2 + z^2}$. To find ∇v use the **chain rule**: $\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-\frac{1}{2}} = 2x \times (-\frac{1}{2}) \times (x^2 + y^2 + z^2)^{-\frac{3}{2}}$ and similarly for the other partial derivatives. Thus

$$\nabla v = -\frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

The above rule therefore gives:

$$\begin{aligned}\nabla^2 f &= 0 + x \times \frac{-2}{\sqrt{x^2 + y^2 + z^2}} + 2\mathbf{i} \cdot \frac{-1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= -\frac{2x}{\sqrt{x^2 + y^2 + z^2}} - \frac{2x^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.\end{aligned}$$

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Solutions to Quizzes

Solution to Quiz: The first and the second order partial derivatives of $f = r^{-n}$, $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ with respect to the variable x read:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial r^{-n}}{\partial r} \times \frac{\partial r}{\partial x} = -nr^{-n-1} \times \frac{x}{r} = -n \frac{x}{r^{n+2}}. \\ \frac{\partial^2 f}{\partial x^2} &= -n \frac{1}{r^{n+2}} \frac{\partial x}{\partial x} - xn \times \frac{\partial}{\partial r} \left(\frac{1}{r^{n+2}} \right) \times \frac{\partial r}{\partial x} \\ &= -n \frac{1}{r^{n+2}} + n(n+2) \frac{x^2}{r^{n+4}}.\end{aligned}$$

Due to the symmetry under interchange of x, y and z we have also

$$\frac{\partial^2 f}{\partial y^2} = -n \frac{1}{r^{n+2}} + n(n+2) \frac{y^2}{r^{n+4}}, \quad \frac{\partial^2 f}{\partial z^2} = -n \frac{1}{r^{n+2}} + n(n+2) \frac{z^2}{r^{n+4}}.$$

Adding these second order derivatives yields the **Laplacian**:

$$\nabla^2 f = -\frac{3n}{r^{n+2}} + n(n+2) \frac{x^2 + y^2 + z^2}{r^{n+4}} = \frac{n(n-1)}{r^{n+2}}.$$

End Quiz

Solution to Quiz:

The Laplacian of the vector field $\mathbf{F} = x^3y\mathbf{i} + \ln(z)\mathbf{j} + \ln(xy)\mathbf{k}$ is a vector $\nabla^2\mathbf{F}$ whose \mathbf{i} , \mathbf{j} components are correspondingly

$$\nabla^2(x^3y) = \frac{\partial^2}{\partial x^2}(x^3y) + \frac{\partial^2}{\partial y^2}(x^3y) + \frac{\partial^2}{\partial z^2}(x^3y) = 6xy,$$

$$\nabla^2(\ln(z)) = \frac{\partial^2}{\partial z^2}(\ln(z)) = \frac{\partial}{\partial z}\left(\frac{1}{z}\right) = -\frac{1}{z^2},$$

while the \mathbf{k} component is

$$\nabla^2(\ln(xy)) = \nabla^2(\ln(x) + \ln(y)) = -\frac{1}{x^2} - \frac{1}{y^2} = -\frac{x^2 + y^2}{x^2y^2}.$$

So the Laplacian of \mathbf{F} is:

$$\nabla^2\mathbf{F} = 6xy\mathbf{i} - \frac{1}{z^2}\mathbf{j} - \frac{x^2 + y^2}{x^2y^2}\mathbf{k}.$$

End Quiz

Solution to Quiz:

The Laplacian of the vector field $\mathbf{F} = 3x^2z\mathbf{i} - \sin(\pi y)\mathbf{j} + \ln(2x^3)\mathbf{k}$ is a vector $\nabla^2\mathbf{F}$ whose \mathbf{i} , \mathbf{j} and \mathbf{k} components are correspondingly

$$\nabla^2(3x^2z) = \frac{\partial^2}{\partial x^2}(3x^2z) + \frac{\partial^2}{\partial y^2}(3x^2z) + \frac{\partial^2}{\partial z^2}(3x^2z) = 6z,$$

$$\nabla^2(-\sin(\pi y)) = -\frac{\partial^2}{\partial y^2}(\sin(\pi y)) = -\frac{\partial}{\partial y}(\pi \cos(\pi y)) = \pi^2 \sin(\pi y),$$

$$\nabla^2(\ln(2x^3)) = \frac{\partial^2}{\partial x^2}(\ln(2x^3)) = \frac{\partial^2}{\partial x^2}(\ln(2) + 3 \ln(x)) = -3\frac{1}{x^2},$$

Therefore the **Laplacian** of \mathbf{F} is

$$\nabla^2\mathbf{F} = 6z\mathbf{i} + \pi^2 \sin(\pi y)\mathbf{j} - 3\frac{1}{x^2}\mathbf{k},$$

and evaluating it at the point $(1, -2, 1)$ we get

$$\nabla^2\mathbf{F} = 6\mathbf{i} + \pi^2 \sin(-2\pi)\mathbf{j} - 3\mathbf{k} = 6\mathbf{i} - 3\mathbf{k}.$$

End Quiz

Solution to Quiz:

The **Laplacian** of $\mathbf{F} = \ln(y)\mathbf{i} + z^2\mathbf{j} - \sin(2\pi x)\mathbf{k}$ at $(1, 1, \pi)$ is the vector whose \mathbf{i} , \mathbf{j} and \mathbf{k} components are in turn given by:

$$\nabla^2 \ln(y) = \frac{\partial^2 \ln(y)}{\partial x^2} + \frac{\partial^2 \ln(y)}{\partial y^2} + \frac{\partial^2 \ln(y)}{\partial z^2} = 0 - \frac{1}{y^2} + 0,$$

$$\nabla^2 z^2 = \frac{\partial^2 z^2}{\partial x^2} + \frac{\partial^2 z^2}{\partial y^2} + \frac{\partial^2 z^2}{\partial z^2} = 0 + 0 + 2,$$

$$\begin{aligned}\nabla^2 \sin(2\pi x) &= \frac{\partial^2 \sin(2\pi x)}{\partial x^2} + \frac{\partial^2 \sin(2\pi x)}{\partial y^2} + \frac{\partial^2 \sin(2\pi x)}{\partial z^2} \\ &= -4\pi^2 \sin(2\pi x) + 0 + 0,\end{aligned}$$

Therefore we have

$$\nabla^2 \mathbf{F} = -\frac{1}{y^2}\mathbf{i} + 2\mathbf{j} - 4\pi^2 \sin(2\pi x)\mathbf{k}$$

Since $\sin(2\pi) = 0$, we find at $(1, 1, \pi)$ that $\nabla^2 \mathbf{F} = -\mathbf{i} + 2\mathbf{j}$.

End Quiz