PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

Determinants 2

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The aim of this package is to provide a short self assessment programme for students who want to calculate three by three determinants.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

The **determinant of the two by two matrix** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written as follows:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{or} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

It is defined as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\,.$$

See the package **Determinants 1**. Here is a brief revision quiz:

Quiz From the answers below, choose $\begin{vmatrix} x & x-1 \\ x+1 & x \end{vmatrix}$.

(a) 1 (b) $2x^2 - 2$ (c) -1 (d) $2x^2 + 1$

2. Three by Three Determinants

A three by three determinant may be written as:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The element a_{ij} is the unique element located in the i-th row and in the j-th column.
The **minor** M_{ij} of a matrix element a_{ij} is obtained by deleting the i-th row and j-th column and evaluating the resulting determinant.

Example 1 Here are some minors of the matrix $\begin{pmatrix} 2 & 3 & 7 \\ 4 & 0 & 5 \\ 1 & 6 & 8 \end{pmatrix}$

$$\begin{aligned} M_{11} &= \begin{vmatrix} 0 & 5 \\ 6 & 8 \end{vmatrix} = 0 \times 8 - 5 \times 6 = -30 \\ M_{32} &= \begin{vmatrix} 2 & 7 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 7 \times 4 = -18 \end{aligned}$$

EXERCISE 1. Calculate the following **minors** of this matrix (click on the **green** letters for the solutions).

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix}$$

(a) M_{11} (b) M_{12}

(c) M_{21} (d) M_{33}

The **cofactor** A_{ij} of the element a_{ij} is obtained by multiplying the minor by the factor $(-1)^{i+j}$. This factor is always $+1$ or -1.
There is a simple way to see what the sign is. The rule produces:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For example, the top left factor is $+1$ since $(-1)^{1+1} = (-1)^2 = +1$.

Rule for calculating determinants: pick *any* row or column, multiply each element by its cofactor and add up the results.

Example 1 Calculate the following determinant :

$$\begin{vmatrix} 3 & 5 & 2 \\ 5 & 4 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

a) by expanding along the top row;
b) by expanding along the second column.

Solution:
a) expanding along the top row we have:

$$\begin{aligned}
\begin{vmatrix} 3 & 5 & 2 \\ 5 & 4 & 1 \\ 1 & 2 & 3 \end{vmatrix} &= 3 \times \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} - 5 \times \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} \\
&= 3(4 \times 3 - 1 \times 2) - 5(5 \times 3 - 1 \times 1) + 2(5 \times 2 - 4 \times 1) \\
&= 3(12 - 2) - 5(15 - 1) + 2(10 - 4) \\
&= 30 - 70 + 12 = -28\,.
\end{aligned}$$

The minus sign in the top line, i.e., -5, is due to the cofactor!

b) This result is again found by expanding along the second column:

$$\begin{aligned} \begin{vmatrix} 3 & 5 & 2 \\ 5 & 4 & 1 \\ 1 & 2 & 3 \end{vmatrix} &= -5 \times \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} + 4 \times \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} \\ &= -5(5 \times 3 - 1 \times 1) + 4(3 \times 3 - 2 \times 1) - 2(3 \times 1 - 2 \times 5) \\ &= -5(15 - 1) + 4(9 - 2) - 2(3 - 10) \\ &= -70 + 28 + 14 = -28\,. \end{aligned}$$

The signs in the first line are again due to the cofactors.

EXERCISE 2. Calculate the determinant below in the following ways (click on the **green** letters for the solutions).

(a) Expanding along the top row.
(b) Expanding along the first column.
(c) Expanding along the second row.
(d) Expanding along the second column
(e) Expanding along the bottom row.
(f) Expanding along the third column.

$$\begin{vmatrix} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

Quiz From the answers below, choose $\begin{vmatrix} 2 & -1 & 3 \\ 1 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$.

(a) 1 (b) 10 (c) 12 (d) 8

Quiz From the answers below, select $\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$.

(a) -32 (b) -24 (c) -16 (d) 20

Quiz From the answers below, pick $\begin{vmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{vmatrix}$.

(a) $1-2x^2$ (b) $1+2x^2$ (c) 1 (d) x^2+1

3. Rules for Determinants

As shown in the package **Determinants 1**, two by two determinants obey the following rules:

Rule 1: The value of a determinant is unchanged by swapping the rows with the corresponding columns (*transposing* it).

Rule 2: If two rows (or columns) are interchanged then the value of the determinant changes sign. *(A determinant with two identical rows or identical columns must vanish.)*

Rule 3: The value of a determinant is unchanged by adding any multiple of the elements of any row (or column) to the corresponding elements of a different row (or column).

Rule 4: A determinant may be multiplied by a constant by multiplying each element of any one row (or column) by that constant.

All of these rules also **hold for higher determinants**. They are assumed, but not proven, below.

Example 2 From Rule 1, a determinant $|A|$ and the determinant of the **transposed matrix** $|A^T|$ must be identical, thus

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix} \quad \text{and} \quad |A^T| = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 3 & 1 \end{vmatrix} \quad \text{are identical.}$$

To check this, expand $|A|$ along the top row:

$$\begin{aligned} |A| &= 3\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} \\ &= 3(-2-6) - 2(1-3) + (2-(-2)) = -16 \end{aligned}$$

While expanding $|A^T|$ along the top row yields:

$$\begin{aligned} |A^T| &= 3\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ 1 & 3 \end{vmatrix} \\ &= 3(-2-6) - (2-2) + (6+2) = -16 \end{aligned}$$

and, as expected, $|A| = |A^T|$.

Example 3 Consider the following determinant:

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (45-48) - 2(36-42) + 3(32-35) \\ &= -3 + 12 - 9 = 0\,.\end{aligned}$$

This can be seen more easily from **Rule 3** as follows.

First subtract column two from column three:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & 1 \end{vmatrix}$$

Next subtract column one from column two:

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix}$$

The second and third columns are now identical. Thus from **Rule 2** the determinant must be zero.

Exercise 3. Use **Rules 3** to simplify the determinants below. (Click on the **green** letters for the solution).

(a) Add row one to row three in $\begin{vmatrix} 3 & 1 & 7 \\ 6 & 3 & 5 \\ -3 & 6 & -7 \end{vmatrix}$

(b) Add three times column two to column three in $\begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & -12 \\ 5 & 9 & 7 \end{vmatrix}$

(c) Subtract row two from row one in $\begin{vmatrix} 2 & 4 & 7 \\ 2 & 4 & 3 \\ 1 & 5 & 2 \end{vmatrix}$

(d) Subtract twice column three from column one in $\begin{vmatrix} 2 & 4 & 1 \\ 2 & 3 & 1 \\ 1 & 5 & -1 \end{vmatrix}$

Exercise 4. Use **Rules 3** and **2** to show that the determinants below all vanish. (Click on the **green** letters for the solution).

(a) $\begin{vmatrix} 2 & 4 & 17 \\ -1 & -2 & 1 \\ 3 & 6 & 16 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 0 \\ -2 & -4 & 5 \\ 5 & 9 & -10 \end{vmatrix}$ (c) $\begin{vmatrix} -2 & 3 & 7 \\ -5 & -8 & 2 \\ 4 & 1 & -7 \end{vmatrix}$

Example 4 From **Rule 4** any common factor in a row or column may be extracted as follows:

$$\begin{vmatrix} a & 1 & 2 \\ 2a & 4 & 3 \\ a & 7 & 5 \end{vmatrix} = a\begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 3 \\ 1 & 7 & 5 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 3 & 1 & 1 \\ b & 2b & b \\ 7 & 2 & 4 \end{vmatrix} = b\begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 7 & 2 & 4 \end{vmatrix}$$

Exercise 5. Use **Rule 4** to simplify the calculation of the determinants below. (Click on the **green** letters for the solution).

(a) $\begin{vmatrix} 2 & x & -1 \\ 1 & 2x & 1 \\ 1 & -2x & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & 0 & -3 \\ 200 & 300 & 400 \\ 1 & -2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} -3 & -1 & -2 \\ -2 & -2 & -2 \\ -2 & -1 & 1 \end{vmatrix}$

Quiz Use **Rule 4** to select the answer equal to $\begin{vmatrix} 6 & 9 & 3 \\ 27 & 18 & -3 \\ -3 & 12 & 15 \end{vmatrix}$.

(a) $-3\begin{vmatrix} 6 & 9 & -1 \\ 27 & 18 & 1 \\ -3 & 12 & 5 \end{vmatrix}$ (b) $3\begin{vmatrix} 2 & 9 & 3 \\ 9 & 18 & -3 \\ 1 & 12 & 15 \end{vmatrix}$

(c) $3\begin{vmatrix} 6 & 3 & 3 \\ 27 & 6 & -3 \\ -3 & 4 & 15 \end{vmatrix}$ (d) $3\begin{vmatrix} 6 & 9 & 3 \\ 27 & 18 & -3 \\ -1 & 3 & 5 \end{vmatrix}$

EXERCISE 6. Use **Rule 4** to simplify the calculation of the determinants below. (Click on the **green** letters for the solution).

(a) $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 3 & 2 & -1 \\ 4 & -2 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -1 \\ -\frac{1}{7} & \frac{3}{14} & -\frac{1}{14} \\ 1 & -2 & -1 \end{vmatrix}$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. What is the value of the determinant $\begin{vmatrix} 2 & 3 & 1 \\ -2 & 0 & 2 \\ 2 & -1 & 3 \end{vmatrix}$?

(a) 4 (b) -135 (c) 36 (d) -4

2. Choose the determinant $\begin{vmatrix} 10 & x & 9 \\ 20 & -x & 9 \\ 10 & 2x & 18 \end{vmatrix}$.

(a) $90x$ (b) $1-2x^2$ (c) $-180x$ (d) $-1800x$

3. For which value of p does $\begin{vmatrix} 2p & 0 & 1 \\ 1-p & 1 & 0 \\ 0 & 1+p & -1+p \end{vmatrix} = 0$?

(a) $\sqrt{6}$ (b) 0 (c) 1 (d) 2

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)
To calculate the minor M_{11} of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix},$$

cross out the row and column passing through the element $a_{11} = 1$ and calculate the determinant of the remaining 2×2 matrix

$$M_{11} = \det \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix} = 2 \times 4 - 5 \times 0 = 8\,.$$

Thus the minor $M_{11} = 8$.

Click on the **green** square to return

□

Exercise 1(b)
To calculate the minor M_{12} of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix},$$

cross out the row and column passing through the element a_{12} and calculate the determinant of the remaining 2×2 matrix

$$M_{12} = \det \begin{pmatrix} \not{1} & \not{3} & \not{2} \\ 3 & \not{2} & 5 \\ 2 & \not{0} & 4 \end{pmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 3 \times 4 - 5 \times 2 = 2\,.$$

The minor $M_{12} = 2\,$.

Click on the **green** square to return

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Exercise 1(c)

To calculate the minor M_{21} of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix},$$

cross out the row and column passing through the element a_{21} and calculate the determinant of the resulting 2×2 matrix

$$M_{21} = \det \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix} = \begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} = 3 \times 4 - 2 \times 0 = 12\,.$$

The minor $M_{21} = 12$.

Click on the **green** square to return

Exercise 1(d)
To calculate the minor M_{33} of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix},$$

cross out the row and column passing through the element $a_{33} = 4$ and calculate the determinant of the remaining 2×2 matrix

$$M_{33} = \det \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 3 = -7\,.$$

We see that the minor $M_{33} = -7$.

Click on the **green** square to return

□

Exercise 2(a)
Expanding the determinant along the top row we have:

$$\begin{aligned}
\begin{vmatrix} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} &= 3 \times \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - 4 \times \begin{vmatrix} 5 & 1 \\ 0 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix} \\
&= 3 \times (0 \times 3 - 1 \times 2) - 4 \times (5 \times 3 - 1 \times 0) \\
&\qquad + 2 \times (5 \times 2 - 0 \times 0) \\
&= 3 \times (-2) - 4 \times 15 + 2 \times 10 \\
&= -6 - 60 + 20 = -46\,.
\end{aligned}$$

We will regain this result in every other part of this exercise.

Click on the **green** square to return

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Exercise 2(b)
Expanding the determinant along the first column we have:

$$\begin{aligned} \begin{vmatrix} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} &= 3 \times \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - 5 \times \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} + 0 \times \begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} \\ &= 3 \times (0 \times 3 - 1 \times 2) - 5 \times (4 \times 3 - 2 \times 2) + 0 \\ &= 3 \times (-2) - 5 \times 8 \\ &= -6 - 40 = -46\,. \end{aligned}$$

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Exercise 2(c)
Expanding the determinant along the second row we find:

$$\begin{aligned}\begin{vmatrix} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} &= -5 \times \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} + 0 \times \begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} \\ &= -5 \times (4 \times 3 - 2 \times 2) + 0 - 1 \times (3 \times 2 - 4 \times 0) \\ &= -5 \times 8 - 1 \times 6 \\ &= -40 - 6 = -46\,.\end{aligned}$$

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Exercise 2(d)
Expanding the determinant along the second column we have:

$$\begin{aligned}
\left|\begin{array}{ccc} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{array}\right| &= -4 \times \left|\begin{array}{cc} 5 & 1 \\ 0 & 3 \end{array}\right| + 0 \times \left|\begin{array}{cc} 3 & 2 \\ 0 & 3 \end{array}\right| - 2 \times \left|\begin{array}{cc} 3 & 2 \\ 5 & 1 \end{array}\right| \\
&= -4 \times (5 \times 3 - 1 \times 0) + 0 - 2 \times (3 \times 1 - 2 \times 5) \\
&= -4 \times 15 - 2 \times (-7) \\
&= -60 + 14 = -46\,.
\end{aligned}$$

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Exercise 2(e)
Expanding the determinant along the bottom row we obtain:

$$\begin{aligned}
\begin{vmatrix} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} &= 0 \times \begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 3 & 4 \\ 5 & 0 \end{vmatrix} \\
&= 0 - 2 \times (3 \times 1 - 2 \times 5) + 3 \times (3 \times 0 - 4 \times 5) \\
&= -2 \times (-7) + 3 \times (-20) \\
&= 14 - 60 = -46\,.
\end{aligned}$$

Click on the **green** square to return

□

Exercise 2(f)
Expanding the determinant along the third column we have:

$$\begin{aligned}
\left|\begin{array}{ccc} 3 & 4 & 2 \\ 5 & 0 & 1 \\ 0 & 2 & 3 \end{array}\right| &= 2 \times \left|\begin{array}{cc} 5 & 0 \\ 0 & 2 \end{array}\right| - 1 \times \left|\begin{array}{cc} 3 & 4 \\ 0 & 2 \end{array}\right| + 3 \times \left|\begin{array}{cc} 3 & 4 \\ 5 & 0 \end{array}\right| \\
&= 2 \times (5 \times 2 - 0 \times 0) - 1 \times (3 \times 2 - 4 \times 0) \\
&\qquad + 3 \times (3 \times 0 - 4 \times 5) \\
&= 2 \times 10 - 1 \times 6 + 3 \times (-20) \\
&= 20 - 6 - 60 = -46\,.
\end{aligned}$$

This is, of course, our result from every other part of this exercise!

Click on the **green** square to return

□

Exercise 3(a)

If for $\begin{vmatrix} 3 & 1 & 7 \\ 6 & 3 & 5 \\ -3 & 6 & -7 \end{vmatrix}$ we add row one to row three, we get:

$$\begin{vmatrix} 3 & 1 & 7 \\ 6 & 3 & 5 \\ -3+3 & 6+1 & -7+7 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 7 \\ 6 & 3 & 5 \\ 0 & 7 & 0 \end{vmatrix}.$$

This is much easier to calculate as only one element of the bottom row is nonzero. Expanding along the bottom row we get:

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 7 \\ 6 & 3 & 5 \\ 0 & 7 & 0 \end{vmatrix} &= -7 \times \begin{vmatrix} 3 & 7 \\ 6 & 5 \end{vmatrix} = -7 \times (3 \times 5 - 7 \times 6) \\ &= -7 \times (15 - 42) = -7 \times (-27) = 189\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 3(b)

Adding 3 times column two to column three in $\begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & -12 \\ 5 & 9 & 7 \end{vmatrix}$ gives:

$$\begin{vmatrix} 1 & 1 & -3+3\times 1 \\ 2 & 4 & -12+3\times 4 \\ 5 & 9 & 7+3\times 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 5 & 9 & 34 \end{vmatrix}$$

This is much easier to calculate, as only one element of the third column is non-zero:

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 5 & 9 & 34 \end{vmatrix} &= +34 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 34 \times (1 \times 4 - 1 \times 2) \\ &= 34 \times (4-2) \\ &= 68\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 3(c)

Subtracting row two from row one in $\begin{vmatrix} 2 & 4 & 7 \\ 2 & 4 & 3 \\ 1 & 5 & 2 \end{vmatrix}$ gives:

$$\begin{vmatrix} 2-2 & 4-4 & 7-3 \\ 2 & 4 & 3 \\ 1 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 4 \\ 2 & 4 & 3 \\ 1 & 5 & 24 \end{vmatrix}$$

This is now simpler to calculate. Expanding along the top row:

$$\begin{aligned} \begin{vmatrix} 0 & 0 & 4 \\ 2 & 4 & 3 \\ 1 & 5 & 24 \end{vmatrix} &= 4 \times \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = 4 \times (2 \times 5 - 4 \times 1) \\ &= 4 \times (10 - 4) \\ &= 24\,. \end{aligned}$$

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Exercise 3(d)

In $\begin{vmatrix} 2 & 4 & 1 \\ 2 & 3 & 1 \\ 1 & 5 & -1 \end{vmatrix}$ we subtract twice column 3 from column one, giving:

$$\begin{vmatrix} 2-2\times 1 & 4 & 1 \\ 2-2\times 1 & 3 & 1 \\ 1-2\times(-1) & 5 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 1 \\ 0 & 3 & 1 \\ 3 & 5 & -1 \end{vmatrix}$$

Since the first column is now very simple, we expand along it:

$$\begin{aligned} \begin{vmatrix} 0 & 4 & 1 \\ 0 & 3 & 1 \\ 3 & 5 & -1 \end{vmatrix} &= 3 \times \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 3 \times (4 \times 1 - 1 \times 3) \\ &= 3 \times 1 \\ &= 3\,. \end{aligned}$$

Click on the **green** square to return

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Exercise 4(a)
To show that the determinant

$$\begin{vmatrix} 2 & 4 & 17 \\ -1 & -2 & 1 \\ 3 & 6 & 16 \end{vmatrix}$$

vanishes, we add row three to row two

$$\begin{vmatrix} 2 & 4 & 17 \\ -1+3 & -2+6 & 1+16 \\ 3 & 6 & 16 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 17 \\ 2 & 4 & 17 \\ 3 & 6 & 16 \end{vmatrix} = 0\,.$$

Here we have used the property (a consequence of **Rule 2**) that: *any determinant with two identical rows vanishes*.

Click on the **green** square to return

□

Exercise 4(b)

To demonstrate that the determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ -2 & -4 & 5 \\ 5 & 9 & -10 \end{vmatrix}$$

is zero, we add twice row two to row three

$$\begin{vmatrix} 1 & 1 & 0 \\ -2 & -4 & 5 \\ 5+2\times(-2) & 9+2\times(-4) & -10+2\times 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -4 & 5 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \; 0\,.$$

Again this determinant vanishes because the first and bottom rows are identical.

Click on the **green** square to return

□

Exercise 4(c)
To verify that the determinant

$$\begin{vmatrix} -2 & 3 & 7 \\ -5 & -8 & 2 \\ 4 & 1 & -7 \end{vmatrix}$$

vanishes, we subtract twice column one from column two

$$\begin{vmatrix} -2 & 3-2\times(-2) & 7 \\ -5 & -8-2\times(-5) & 2 \\ 4 & 1-2\times 4 & -7 \end{vmatrix} = \begin{vmatrix} -2 & 7 & 7 \\ -5 & 2 & 2 \\ 4 & -7 & -7 \end{vmatrix}$$

$$= 0\,.$$

The determinant must vanish because the second column and the third column are identical.

Click on the **green** square to return

□

Exercise 5(a)

From **Rule 4** we have $\begin{vmatrix} 2 & x & -1 \\ 1 & 2x & 1 \\ 1 & -2x & 1 \end{vmatrix} = x \times \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix}$.

This may be further simplified by subtracting the third column from the first column:

$$
\begin{aligned}
x \times \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} &= x \times \begin{vmatrix} 2-(-1) & 1 & -1 \\ 1-1 & 2 & 1 \\ 1-1 & -2 & 1 \end{vmatrix} \\
&= x \times \begin{vmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 3x \times \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} \\
&= 3x \times \{2 \times 2 - 1 \times (-2)\} \\
&= 12x\,.
\end{aligned}
$$

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Exercise 5(b)

From **Rule 4** we have $\begin{vmatrix} 3 & 0 & -3 \\ 200 & 300 & 400 \\ 1 & -2 & 1 \end{vmatrix} = 100 \times \begin{vmatrix} 3 & 0 & -3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}$.

Adding the first column to the third column further simplifies the calculations:

$$\begin{aligned}
100 \times \begin{vmatrix} 3 & 0 & -3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} &= 100 \times \begin{vmatrix} 3 & 0 & -3+3 \\ 2 & 3 & 4+2 \\ 1 & -2 & 1+1 \end{vmatrix} \\
&= 100 \times \begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 6 \\ 1 & -2 & 2 \end{vmatrix} = 100 \times 3 \times \begin{vmatrix} 3 & 6 \\ -2 & 2 \end{vmatrix} \\
&= 300 \times (3 \times 2 - 6 \times (-2)) = 300 \times 18 \\
&= 5400\,.
\end{aligned}$$

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Exercise 5(c)

To simplify the calculation of the determinant $|A| = \begin{vmatrix} -3 & -1 & -2 \\ -2 & -2 & -2 \\ -2 & -1 & 1 \end{vmatrix}$

extract factors of $(-1)\,,(-2)$ and (-1) from the first, second and third rows respectively

$$\begin{aligned} |A| &= (-1)\times(-2)\times(-1)\times\begin{vmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\ &= -2\times\left\{3\times\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 2\times\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\right\} \\ &= -2\,\{3(1\times(-1)-1\times 1)-(1\times(-1)-1\times 2) \\ &\qquad + 2\times(1\times 1-1\times 2)\} = -2(-6+3-2) \\ &= 10\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(a)

Extracting the common factor of 1/2 from the first row of the determinant gives:

$$|A| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 3 & 2 & -1 \\ 4 & -2 & 3 \end{vmatrix}$$

$$\begin{aligned}
|A| &= \frac{1}{2} \times \begin{vmatrix} 1 & 1 & -1 \\ 3 & 2 & -1 \\ 4 & -2 & 3 \end{vmatrix} \\
&= \frac{1}{2} \times \left\{ \begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 4 & -2 \end{vmatrix} \right\} \\
&= \frac{1}{2} \times \{(2 \times 3 - (-1) \times (-2)) - (3 \times 3 - (-1) \times 4) \\
&\qquad - (3 \times (-2) - 2 \times 4)\} \\
&= \frac{1}{2} \times (4 - 13 + 12) = \frac{3}{2}\,.
\end{aligned}$$

Click on the **green** square to return □

Exercise 6(b)

Extract the common factor of $1/14$ from the second row of the determinant

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ -\frac{1}{7} & \frac{3}{14} & -\frac{1}{14} \\ 1 & -2 & -1 \end{vmatrix}$$

$$\begin{aligned}
|A| &= \frac{1}{14} \times \begin{vmatrix} 2 & 1 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & -1 \end{vmatrix} \\
&= \frac{1}{14} \times \left\{ - \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} \right\} \\
&= \frac{1}{14} \times \{-((-2) \times (-2) - 3 \times 1) + (2 \times (-2) - 1 \times 1) \\
&\qquad - (2 \times 3 - 1 \times (-2))\} \\
&= \frac{1}{14} \times (-1 - 5 - 8) = -\frac{14}{14} = -1\,.
\end{aligned}$$

Click on the **green** square to return ☐

Solutions to Quizzes

Solution to Quiz:
Using the definition of 2×2 determinants, we have

$$\begin{aligned}\begin{vmatrix} x & x-1 \\ x+1 & x \end{vmatrix} &= x \times x - (x-1) \times (x+1) \\ &= x^2 - (x^2 + x - x - 1) \\ &= x^2 - x^2 + 1 \\ &= 1\,.\end{aligned}$$

Hence

$$\det \begin{pmatrix} x & x-1 \\ x+1 & x \end{pmatrix} = 1\,.$$

End Quiz

Solution to Quiz:
To calculate the determinant, let us expand it along the first row:

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 3 \\ 1 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix} &= 2 \times \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} - (-1) \times \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \\
&\quad + 3 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \\
&= 2 \times (-1 \times 1 - (-2) \times 2) \\
&\quad\quad +1 \times (1 \times 1 - (-2) \times (-1)) \\
&\quad\quad\quad +3 \times (1 \times 2 - (-1) \times (-1)) \\
&= 2 \times (-1 + 4) + 1 \times (1 - 2) + 3 \times (2 - 1) \\
&= 6 - 1 + 3 \\
&= 8\,.
\end{aligned}$$

End Quiz

Solution to Quiz:
To calculate the determinant, let us expand it along the first column:

$$\begin{aligned}
\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & 1 \end{vmatrix} &= 3 \times \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} \\
&= 3 \times (-2 \times 1 - 3 \times 2) - 1 \times (2 \times 1 - 1 \times 2) \\
&\qquad +1 \times (2 \times 3 - 1 \times (-2)) \\
&= 3 \times (-2 - 6) + 1 \times (2 - 2) + 1 \times (6 + 2) \\
&= -24 + 8 \\
&= -16 \,.
\end{aligned}$$

Note that $\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}$ must *vanish as the rows are identical*, see the package **Determinants 1** and also later in this package.

End Quiz

Solution to Quiz:
To calculate the determinant, let us expand it along the first column:

$$\begin{aligned}
\begin{vmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{vmatrix} &= 1 \times \begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} - x \times \begin{vmatrix} x & 0 \\ x & 1 \end{vmatrix} + 0 \times \begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix} \\
&= (1 \times 1 - x \times x) - x \times (x \times 1 - 0 \times x) + 0 \\
&= (1 - x^2) - x \times x \\
&= 1 - 2x^2\,.
\end{aligned}$$

End Quiz

Solution to Quiz:
Consider the given determinant

$$|A| = \begin{vmatrix} 6 & 9 & 3 \\ 27 & 18 & -3 \\ -3 & 12 & 15 \end{vmatrix}$$

and extract the common factor of 3 from the second column

$$\begin{aligned} |A| &= \begin{vmatrix} 6 & 9 & 3 \\ 27 & 18 & -3 \\ -3 & 12 & 15 \end{vmatrix} \\ &= \begin{vmatrix} 6 & 3 \times 3 & 3 \\ 27 & 3 \times 6 & -3 \\ -3 & 3 \times 4 & 15 \end{vmatrix} = 3 \times \begin{vmatrix} 6 & 3 & 3 \\ 27 & 6 & -3 \\ -3 & 4 & 15 \end{vmatrix} . \end{aligned}$$

End Quiz