Basic Mathematics

Determinants 1

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The aim of this package is to provide a short self assessment programme for students who want to calculate simple determinants.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

Consider the following set of simultaneous equations:

$$\begin{aligned} ax + by &= p \qquad (1) \\ cx + dy &= q \qquad (2) \end{aligned}$$

The variable y can be eliminated from the equations by multiplying equation (1) by d and equation (2) by b. (See the package on **Simultaneous Equations**.) This leads to the new equations:

$$\begin{aligned} adx + bdy &= pd \qquad (3) \\ bcx + bdy &= bq \qquad (4) \end{aligned}$$

Subtracting (4) from (3) gives:

$$(ad - bc)x = pd - bq$$

In a similar way, eliminating x, leads to:

$$(ad - bc)y = qa - pc$$

The **same important factor** $ad - bc$ appears on the left in each case.

If the factor $ad - bc$ *vanishes*, then the two simultaneous equations for x and y cannot be solved since one would have:

$$(ad-bc)x = pd - bq \quad \Rightarrow \quad 0 \times x = pd - bq$$
$$(ad-bc)y = qa - pc \quad \Rightarrow \quad 0 \times y = qa - pc$$

To see what is happening, *assume* $ad = bc$, so $a = \frac{bc}{d}$ and our initial set of simultaneous equations (1) and (2) become:

$$\frac{bc}{d}x + by = p \qquad (5)$$
$$cx + dy = q \qquad (6)$$

If we now multiply (5) by d it becomes

$$bcx + bdy = pd$$
$$b(cx + dy) = pd$$
$$\therefore cx + dy = \frac{pd}{b}$$

The left hand side of this equation is the same as the left side of (6).

As far as the right hand sides of (5) and (6) are concerned, there are two possibilities. Either $pd/b = q$ in which case equations (5) and (6) are the same (and are said not to be **independent**) or $pd/b \neq q$ in which case equations (5) and (6) are **inconsistent** since their left hand sides are the same but their right hand sides disagree.

An example of non-independent equations would be:

$$3x + 2y = 4 \qquad (7)$$
$$6x + 4y = 8 \qquad (8)$$

Since equation (8) is simply twice equation (7). The factor $ad - bc$ is here $3 \times 4 - 2 \times 6 = 12 - 12 = 0$.

An example of inconsistent equations (no possible solutions) is

$$4x + 5y = 3 \qquad (9)$$
$$4x + 5y = 7 \qquad (10)$$

Again the factor $ad - bc$ vanishes: $4 \times 5 - 5 \times 4 = 20 - 20 = 0$.

This important factor $ad - bc$ is the subject of this package.

The initial equations may be written in matrix form as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} .$$

The factor $ad-bc$ is called the **determinant** of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Notation: The determinant is written as $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

This is the **determinant of a two by two matrix** made up from:

$$\begin{vmatrix} a & b \\ \cdot & \cdot \end{vmatrix} \qquad \begin{vmatrix} \cdot & \cdot \\ c & d \end{vmatrix} \qquad \begin{vmatrix} a & \cdot \\ c & \cdot \end{vmatrix} \qquad \begin{vmatrix} \cdot & b \\ \cdot & d \end{vmatrix}$$

first row   second row   first column   second column

Determinants are important in mathematics and they have extensive applications in science and engineering.

2. Two by Two Determinants

A two by two determinant is defined by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\,.$$

Example 1 Calculate the following determinant:

$$\begin{vmatrix} 3 & 6 \\ 5 & 4 \end{vmatrix} = 3 \times 4 - 6 \times 5 = 12 - 30 = -18\,.$$

Exercise 1. Calculate the following determinants (click on the **green** letters for the solutions).

(a) $\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 8 & 10 \\ 4 & 7 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 5 & 7 \\ 8 & -9 \end{vmatrix}$

(e) $\begin{vmatrix} -9 & 8 \\ -8 & 9 \end{vmatrix}$ (f) $\begin{vmatrix} 2 & 3 \\ -1 & 7 \end{vmatrix}$ (g) $\begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix}$ (h) $\begin{vmatrix} 3 & 6 \\ 5 & 4 \end{vmatrix}$

Here are some determinants involving symbols:

Quiz From the answers below, select the determinant of $\begin{vmatrix} x & 2x \\ 3 & 4 \end{vmatrix}$.

(a) $8x$ (b) $-2x$ (c) $2x$ (d) $10x$

Here are some more exercises:

Exercise 2. Calculate the following determinants (click on the **green** letters for the solutions).

(a) $\begin{vmatrix} x^2 & 5x \\ x & 3 \end{vmatrix}$ (b) $\begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix}$ (c) $\begin{vmatrix} w & 3w+1 \\ -1 & -3w \end{vmatrix}$

(d) $\begin{vmatrix} t & t \\ -2t & 2t \end{vmatrix}$ (e) $\begin{vmatrix} x & x \\ x+1 & x+1 \end{vmatrix}$ (f) $\begin{vmatrix} x^{-2} & x \\ 1 & x^3 \end{vmatrix}$

3. Properties of Determinants

Rule 1: The value of a determinant is unchanged by interchanging the rows with the corresponding columns.

Proof: Consider the matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

The matrix A^T is found by replacing the first row of A by the first column and the second row with the second column. We call A^T the **transpose** of the matrix A.
It may be checked that $\det A = \det A^T = ad - bc$.

This shows that any statement made for the rows of a determinant also holds for columns!

Exercise 3. Swap the rows and columns in the matrices below and check that the determinants of the matrices are not changed (click on the **green** letters for the solutions).

(a) $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -5 \\ 4 & 10 \end{pmatrix}$ (c) $\begin{pmatrix} w & w+1 \\ -w+1 & -3w \end{pmatrix}$

Rule 2: If two rows or columns are interchanged then the value of the determinant changes sign.

Proof: We know that:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If we swap the columns around, we get

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad = -(ad - bc) = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} .$$

EXERCISE 4. (Click on the **green** letter for the solution).

(a) Repeat the above proof, but now swap the rows rather than the columns.

(b) Swap both rows and columns to show that $\begin{vmatrix} d & c \\ b & a \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

An important consequence of **Rule 2** is that **any determinant must be equal to zero if two of its rows or two of its columns are identical**.

Example 3: a) Let us check that the following determinant with two identical columns vanishes:

$$\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix} = 3 \times 5 - 3 \times 5 = 0$$

b) Similarly the determinant below with two identical rows vanishes:

$$\begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} = 2 \times (-1) - (-1) \times 2 = 0$$

Quiz From the property above, for which x does $\begin{vmatrix} -5 & 3 \\ 5-x & 3 \end{vmatrix}$ vanish?

(a) 0 (b) 10 (c) 2 (d) 5

Rule 3: The value of a determinant is unchanged by adding any multiple of the elements of any row (or column) to the corresponding elements of a different row (or column).

Proof: Below we add n times the first column to the second column:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow \begin{vmatrix} a & b+na \\ c & d+nc \end{vmatrix}$$

It can be seen that the determinant is unchanged:

$$\begin{aligned} \begin{vmatrix} a & b+na \\ c & d+nc \end{vmatrix} &= a(d+nc)-(b+na)d \\ &= ad+anc-bc-nac = ad-bc \end{aligned}$$

EXERCISE 5. (Click on the **green** letter for the solution).

(a) Repeat the above proof, but now add n times row 2 to row 1.

(b) *Use the above rule* to simplify the calculation of: $\begin{vmatrix} 14 & -13 \\ 8 & -8 \end{vmatrix}$

(c) *Use the above rule* to simplify: $\begin{vmatrix} 3-a & a \\ 4a & -4a \end{vmatrix}$

Rule 4: A determinant may be multiplied by a constant by multiplying each element of any one row (or column) by that constant.

Proof:

$$n\begin{vmatrix} a & c \\ b & d \end{vmatrix} = n(ad - bc) = adn - bcn$$

is just the same as

$$\begin{vmatrix} an & c \\ bn & d \end{vmatrix} = adn - bcn \quad \text{and also} \quad \begin{vmatrix} a & c \\ bn & dn \end{vmatrix} = adn - bcn$$

This rule is **used** to simplify the calculation of determinants.

Exercise 6. Use the above **rule** to extract common factors in the determinants below:

(a) $\begin{vmatrix} 1000 & 2 \\ 2000 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -125 & -25 \\ 1 & 7 \end{vmatrix}$ (c) $\begin{vmatrix} 8 & 64 \\ 7 & 49 \end{vmatrix}$

(d) $\begin{vmatrix} t^5 & 3 \\ 3t^4 & 9 \end{vmatrix}$ (e) $\begin{vmatrix} ax^3 & x^2 \\ 1 & a \end{vmatrix}$ (f) $\begin{vmatrix} (x+2)^6 & x^{-1} \\ (x+2)^7 & x^{-1} \end{vmatrix}$

For reference, here are all the rules again:

Summary:

Rule 1: The value of a determinant is unchanged by swapping the rows with the corresponding columns.

Rule 2: If two rows or columns are interchanged then the value of the determinant changes sign.

Rule 3: The value of a determinant is unchanged by adding any multiple of the elements of any row (or column) to the corresponding elements of a different row (or column).

Rule 4: A determinant may be multiplied by a constant by multiplying each element of any one row (or column) by that constant.

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. Choose the determinant $\begin{vmatrix} 17 & -4 \\ -21 & 3 \end{vmatrix}$.

(a) 135 (b) -135 (c) -33 (d) 33

2. Select the determinant $\begin{vmatrix} (1-x) & -x \\ x & (1+x) \end{vmatrix}$.

(a) $1-2x$ (b) $1-2x^2$ (c) 0 (d) 1

3. For which of the values of m below does $\begin{vmatrix} 3m & 6 \\ 8 & m \end{vmatrix}$ vanish?

(a) $\sqrt{48}$ (b) 0 (c) -4 (d) 2

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)
The determinant of the matrix

$$\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$$

is evaluated as follows:

$$\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 5 \times 3 = 8 - 15 = -7\,.$$

Click on the **green** square to return

□

Exercise 1(b)
The determinant of the matrix

$$\begin{pmatrix} 8 & 10 \\ 4 & 7 \end{pmatrix}$$

is evaluated below:

$$\begin{vmatrix} 8 & 10 \\ 4 & 7 \end{vmatrix} = 8 \times 7 - 10 \times 4 = 56 - 40 = 16\,.$$

Click on the **green** square to return

□

Exercise 1(c)
The determinant of the matrix

$$\begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$$

is calculated below:

$$\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 3 \times 2 - (-2) \times 1 = 6 + 2 = 8\,.$$

Click on the **green** square to return

□

Exercise 1(d)
The determinant of the matrix

$$\begin{pmatrix} 5 & 7 \\ 8 & -9 \end{pmatrix}$$

is given by

$$\begin{vmatrix} 5 & 7 \\ 8 & -9 \end{vmatrix} = 5 \times (-9) - 7 \times 8 = -45 - 56 = -101\,.$$

Click on the **green** square to return

□

Exercise 1(e)
The determinant of the matrix

$$\begin{pmatrix} -9 & 8 \\ -8 & 9 \end{pmatrix}$$

is given by

$$\begin{vmatrix} -9 & 8 \\ -8 & 9 \end{vmatrix} = (-9) \times 9 - 8 \times (-8) = -81 + 64 = -17\,.$$

Click on the **green** square to return

□

Exercise 1(f)
The determinant of the matrix

$$\begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix}$$

is found as follows:

$$\begin{vmatrix} 2 & 3 \\ -1 & 7 \end{vmatrix} = 2 \times 7 - 3 \times (-1) = 14 + 3 = 17 \, .$$

Click on the **green** square to return

□

Exercise 1(g)
A direct calculation of the determinant of the matrix

$$\begin{pmatrix} -2 & 2 \\ 3 & -4 \end{pmatrix}$$

gives

$$\begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix} = (-2) \times (-4) - 2 \times 3 = 8 - 6 = 2\,.$$

Click on the **green** square to return

□

Exercise 1(h)
A direct calculation of the determinant of the matrix

$$\begin{pmatrix} 3 & 6 \\ 5 & 4 \end{pmatrix}$$

gives

$$\begin{vmatrix} 3 & 6 \\ 5 & 4 \end{vmatrix} = 3 \times 4 - 6 \times 5 = 12 - 30 = -18\,.$$

Click on the **green** square to return

□

Exercise 2(a)

Evaluation of the determinant $\det \mathrm{A}$ of the matrix

$$\mathrm{A} = \begin{pmatrix} x^2 & 5x \\ x & 3 \end{pmatrix}$$

yields

$$\begin{aligned} \det \mathrm{A} = \begin{vmatrix} x^2 & 5x \\ x & 3 \end{vmatrix} &= x^2 \times 3 - 5x \times x \\ &= 3x^2 - 5x^2 \\ &= -2x^2 \,. \end{aligned}$$

Thus $\det \mathrm{A} = -2x^2$.

Click on the **green** square to return

□

Exercise 2(b)
Calculation of the determinant $\det \mathrm{A}$ of the matrix

$$\mathrm{A} = \begin{pmatrix} x-1 & -1 \\ 1 & x+1 \end{pmatrix}$$

gives

$$\begin{aligned} \det \mathrm{A} = \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} &= (x-1)\times(x+1)-(-1)\times 1 \\ &= x^2+x-x-1+1 \\ &= x^2 . \end{aligned}$$

Hence $\det \mathrm{A} = x^2$.

Click on the **green** square to return

□

Exercise 2(c)
The determinant $\det A$ of the matrix

$$A = \begin{pmatrix} w & 3w+1 \\ -1 & -3w \end{pmatrix}$$

is

$$\begin{aligned} \det A = \begin{vmatrix} w & 3w+1 \\ -1 & -3w \end{vmatrix} &= w \times (-3w) - (3w+1) \times (-1) \\ &= -3w^2 + 3w + 1\,. \end{aligned}$$

Therefore $\det A = -3w^2 + 3w + 1\,.$

Click on the **green** square to return

□

Exercise 2(d)

The determinant $\det \mathrm{M}$ of the matrix

$$\mathrm{M} = \begin{pmatrix} t & t \\ -2t & 2t \end{pmatrix}$$

is

$$\begin{aligned} \det \mathrm{M} = \begin{vmatrix} t & t \\ -2t & 2t \end{vmatrix} &= t \times 2t - t \times (-2t) \\ &= 2t^2 + 2t^2 \\ &= 4t^2 \,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 2(e)
The determinant $\det \mathrm{A}$ of the matrix

$$\mathrm{A} = \begin{pmatrix} x & x \\ x+1 & x+1 \end{pmatrix}$$

is

$$\begin{aligned} \det \mathrm{A} = \begin{vmatrix} x & x \\ x+1 & x+1 \end{vmatrix} &= x \times (x+1) - x \times (x+1) \\ &= 0\,. \end{aligned}$$

We see that $\det \mathrm{A}$ is identically zero for all values of x .
Click on the **green** square to return

□

Exercise 2(f)
The evaluation of the determinant $\det A$ of the matrix

$$A = \begin{pmatrix} x^{-2} & x \\ 1 & x^3 \end{pmatrix}$$

gives

$$\begin{aligned} \det A = \begin{vmatrix} x^{-2} & x \\ 1 & x^3 \end{vmatrix} &= x^{-2} \times x^3 - x \times 1 \\ &= x^{-2+3} - x \\ &= x - x \\ &= 0\,. \end{aligned}$$

Here we used $x^a x^b = x^{a+b}$, see the package on **Powers**.

Click on the **green** square to return

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Exercise 3(a)

The determinant of the matrix $\mathrm{A} = \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ is

$$\det \mathrm{A} = \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3 \times 2 - 4 \times (-1) = 6 + 4 = 10\,.$$

Interchanging the rows of the matrix A with the corresponding columns we obtain the new matrix A^{T}, the transpose of the matrix A:

$$\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

whose determinant is

$$\det \mathrm{A}^{\mathrm{T}} = \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = 3 \times 2 - (-1) \times 4 = 6 + 4 = 10\,.$$

Therefore we see that $\det \mathrm{A}^{\mathrm{T}} = \det \mathrm{A}$.

Click on the **green** square to return

□

Exercise 3(b)

Calculating the determinant of the matrix $\mathrm{A} = \begin{pmatrix} 2 & -5 \\ 4 & 10 \end{pmatrix}$ we have

$$\det \mathrm{A} = \begin{vmatrix} 2 & -5 \\ 4 & 10 \end{vmatrix} = 2 \times 10 - (-5) \times 4 = 20 + 20 = 40\,.$$

Interchanging the rows of the matrix A with the corresponding columns we obtain the new matrix A^{T}, the transpose of the matrix A,

$$\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ -5 & 10 \end{pmatrix}$$

whose determinant is

$$\det \mathrm{A}^{\mathrm{T}} = \begin{vmatrix} 2 & 4 \\ -5 & 10 \end{vmatrix} = 2 \times 10 - 4 \times (-5) = 20 + 20 = 40\,.$$

We see that $\det \mathrm{A}^{\mathrm{T}} = \det \mathrm{A}$.

Click on the **green** square to return

□

Exercise 3(c)

The determinant of the matrix $\mathrm{A} = \begin{pmatrix} w & w+1 \\ -w+1 & -3w \end{pmatrix}$ is

$$\begin{aligned} \det \mathrm{A} &= \begin{vmatrix} w & w+1 \\ -w+1 & -3w \end{vmatrix} = w \times (-3w) - (w+1) \times (-w+1) \\ &= -3w^2 + w^2 - w + w - 1 = -2w^2 - 1 \,. \end{aligned}$$

Constructing the transpose of the matrix A

$$\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} w & -w+1 \\ w+1 & -3w \end{pmatrix}$$

and calculating its determinant we find

$$\begin{aligned} \det \mathrm{A}^{\mathrm{T}} &= \begin{vmatrix} w & -w+1 \\ w+1 & -3w \end{vmatrix} = w \times (-3w) - (-w+1) \times (w+1) \\ &= -3w^2 + w^2 + w - w - 1 = -2w^2 - 1 = \det \mathrm{A} \,. \end{aligned}$$

Click on the **green** square to return ☐

Exercise 4(a)
The determinant of the two by two matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\,.$$

If the rows are interchanged, we have

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

i.e., the sign of the determinant changes.

Note that we did not really need to prove this separately, as from **Rule 1** we know that if it holds for columns it must hold for rows.

Click on the **green** square to return

□

Exercise 4(b)

For the two by two matrix $\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose determinant is $\det \mathrm{A} = ab - cd$ let us swap the columns

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

and then swap the rows

$$\begin{pmatrix} b & a \\ d & c \end{pmatrix} \rightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

The determinant of the resulting matrix is

$$\begin{vmatrix} d & c \\ b & a \end{vmatrix} = da - cb = ad - bc = \det \mathrm{A}\,.$$

which is the determinant of the original matrix A.
Note that since each swap changes the sign of the determinant, two swaps introduce a factor of $(-1)^2 = +1$.

Click on the **green** square to return □

Exercise 5(a)

To see that adding n times the second row to the first row does not change the determinant, consider

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \rightarrow \quad \begin{vmatrix} a+nc & b+nd \\ c & d \end{vmatrix}$$

The determinant is

$$\begin{vmatrix} a+nc & b+nd \\ c & d \end{vmatrix} = (a+nc)d-(b+nd)c$$
$$= ad+ncd-bc-ndc = ad-bc\,,$$

which is the original value.

Note that we did not really need to prove this separately, as from **Rule 1** we know that if it holds for columns it must hold for rows.

Click on the **green** square to return

□

Exercise 5(b)
To simplify the calculation of the determinant

$$\begin{vmatrix} 14 & -13 \\ 8 & -8 \end{vmatrix}$$

we add the second column to the first one

$$\begin{aligned} \begin{vmatrix} 14+(-13) & -13 \\ 8+(-8) & -8 \end{vmatrix} &= \begin{vmatrix} 1 & -13 \\ 0 & -8 \end{vmatrix} \\ &= 1\times(-8)+0 \\ &= -8\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 5(c)
To simplify the calculation of the determinant

$$\begin{vmatrix} 3-a & a \\ 4a & -4a \end{vmatrix}$$

we add the second column to the first one

$$\begin{aligned} \begin{vmatrix} 3-a+(a) & a \\ 4a+(-4a) & -4a \end{vmatrix} &= \begin{vmatrix} 3 & a \\ 0 & -4a \end{vmatrix} \\ &= 3\times(-4a)-0 \\ &= -12a\,. \end{aligned}$$

Click on the **green** square to return ☐

Exercise 6(a)

To simplify the calculation of $\begin{vmatrix} 1000 & 2 \\ 2000 & 3 \end{vmatrix}$ let us extract the common factor of 1000 from the first column:

$$\begin{aligned} \begin{vmatrix} 1000 & 2 \\ 2000 & 3 \end{vmatrix} = \begin{vmatrix} 1 \times 1000 & 2 \\ 2 \times 1000 & 3 \end{vmatrix} &= 1000 \times \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \\ &= 1000 \times (1 \times 3 - 2 \times 2) \\ &= 1000 \times (-1) \\ &= -1000\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(b)

The evaluation of $\begin{vmatrix} -125 & -25 \\ 1 & 7 \end{vmatrix}$ can be simplified by extracting the common factor of (-25) from the first row:

$$\begin{aligned} \begin{vmatrix} -125 & -25 \\ 1 & 7 \end{vmatrix} &= \begin{vmatrix} (-25)\times 5 & (-25)\times 1 \\ 1 & 7 \end{vmatrix} \\ &= -25 \times \begin{vmatrix} 5 & 1 \\ 1 & 7 \end{vmatrix} \\ &= -25 \times (5\times 7 - 1\times 1) \\ &= -25 \times 34 \\ &= -850\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(c)

The evaluation of $\begin{vmatrix} 8 & 64 \\ 7 & 49 \end{vmatrix}$ can be simplified by extracting two common factors: 8 from the first row and also 7 from the second row:

$$\begin{aligned} \begin{vmatrix} 8 & 64 \\ 7 & 49 \end{vmatrix} &= \begin{vmatrix} 8\times 1 & 8\times 8 \\ 7\times 1 & 7\times 7 \end{vmatrix} = 8\times 7\times \begin{vmatrix} 1 & 8 \\ 1 & 7 \end{vmatrix} \\ &= 56\times(1\times 7 - 8\times 1) \\ &= 56\times(-1) \\ &= -56\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(d)

To find the determinant $\begin{vmatrix} t^5 & 3 \\ 3t^4 & 9 \end{vmatrix}$ one can extract two common factors: t^4 from the first column and also 3 from the second column:

$$\begin{vmatrix} t^5 & 3 \\ 3t^4 & 9 \end{vmatrix} = \begin{vmatrix} t^4 \times t & 3 \times 1 \\ t^4 \times 3 & 3 \times 3 \end{vmatrix} = t^4 \times 3 \times \begin{vmatrix} t & 1 \\ 3 & 3 \end{vmatrix}$$

Now we can again extract the factor of 3 from the second row:

$$\begin{aligned} 3t^4 \times \begin{vmatrix} t & 1 \\ 3 & 3 \end{vmatrix} &= 3t^4 \times 3 \times \begin{vmatrix} t & 1 \\ 1 & 1 \end{vmatrix} \\ &= 9t^4 \times (t \times 1 - 1 \times 1) \\ &= 9t^4(t-1)\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(e)

To find the determinant $\begin{vmatrix} ax^3 & x^2 \\ 1 & a \end{vmatrix}$ one can extract the common factor of x^2 from the first row:

$$\begin{aligned} \begin{vmatrix} ax^3 & x^2 \\ 1 & a \end{vmatrix} &= \begin{vmatrix} x^2 \times ax & x^2 \times 1 \\ 1 & a \end{vmatrix} = x^2 \times \begin{vmatrix} ax & 1 \\ 1 & a \end{vmatrix} \\ &= x^2 \times (ax \times a - 1 \times 1) \\ &= x^2 \times (a^2 x - 1) \\ &= x^2 (a^2 x - 1)\,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 6(f)

To find the determinant $\begin{vmatrix} (x+2)^6 & x^{-1} \\ (x+2)^7 & x^{-1} \end{vmatrix}$ one can extract the common factor of $(x+2)^6$ from the first column and the factor x^{-1} from the second column:

$$\begin{aligned} \begin{vmatrix} (x+2)^6 & x^{-1} \\ (x+2)^7 & x^{-1} \end{vmatrix} &= (x+2)^6 \times x^{-1} \times \begin{vmatrix} 1 & 1 \\ x+2 & 1 \end{vmatrix} \\ &= \frac{(x+2)^6}{x} \times (1 \times 1 - 1 \times (x+2)) \\ &= \frac{(x+2)^6}{x} \times (1 - x - 2) \\ &= -\frac{(x+2)^6(x+1)}{x} . \end{aligned}$$

Click on the **green** square to return

□

Solutions to Quizzes

Solution to Quiz:

The evaluation of the determinant $\det \mathrm{A}$ of the matrix $\mathrm{A} = \begin{pmatrix} x & 2x \\ 3 & 4 \end{pmatrix}$ yields

$$\det \mathrm{A} = \begin{vmatrix} x & 2x \\ 3 & 4 \end{vmatrix} = x \times 4 - 2x \times 3 = 4x - 6x = -2x\,.$$

Hence $\det \mathrm{A} = -2x$.

End Quiz

Solution to Quiz:
The determinant of the matrix

$$A = \begin{pmatrix} -5 & 3 \\ 5-x & 3 \end{pmatrix}$$

will vanish when the rows are identical, i.e. when

$$-5 = 5 - x\,.$$

This happens for

$$x = 10\,.$$

This result may be *checked* by calculating the determinant:

$$\det A = -15 - 3(5-x) = -30 + 3x\,.$$

For this to vanish, $-30 + 3x = 0$, we need $x = 10$.

End Quiz