

Introduction to Matrices

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of matrices, their addition and subtraction and elementary row operations.

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1. Matrices (Introduction)

A matrix is a rectangular array of numbers.

Example 1 Each of the following are examples of matrices.

$$A = \begin{pmatrix} 4 & 3 & -1 \\ 8 & -0.5 & 34 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -3 & 3 & -7 \end{pmatrix},$$
$$C = \begin{pmatrix} 5 & -3 & 7 \\ 7 & 0 & -7 \\ 0 & 25 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -5 \\ 4 \\ -57 \\ 34 \end{pmatrix}$$

The matrix A has **two** rows and **three** columns, it is a 2×3 (read as “**two** by **three**”), matrix.

The matrix B has **one** row and **four** columns, it is a 1×4 matrix.

The matrix C has **three** rows and **three** columns, it is a 3×3 matrix.

The matrix D has **four** rows and **one** column, it is a 4×1 matrix.

The matrix B is also called a **row vector** whilst the matrix D is called a **column vector**.

The elements of a matrix are referred to (or labelled) by their **row number** and **column number**, but *always in that order*, i.e. the row number followed by the column number. The ij element of a matrix X , often written as x_{ij} , is the element in the i -th row and j -th column.

Example 2 The matrix X is written below.

- (a) What is the element x_{21} ?
- (b) What is the element x_{32} ?
- (c) What is the label for 5?
- (d) What is the label for 9?

$$X = \begin{pmatrix} -3 & 4 & 7 & -2 \\ -4 & -6 & 21 & 5 \\ -7 & 0 & 8 & 9 \end{pmatrix}$$

Solution

- (a) The element x_{21} is -4 .
- (b) The element x_{32} is 0 .
- (c) 5 is the element x_{24} .
- (d) 9 is the element x_{34} .

Quiz What is the label for -7 in the matrix C of **example 1**?

- (a) c_{32} ,
- (b) c_{23} ,
- (c) c_{13} ,
- (d) c_{21} .

2. Addition of Matrices

Two matrices, A and B , may be added together provided that they have the same number of rows and the same number of columns. If A and B are both $m \times n$ matrices then $A + B$ exists and is also a $m \times n$ matrix. If a_{ij} and b_{ij} are the ij elements of A and B respectively, then $a_{ij} + b_{ij}$ is the ij element of $A + B$.

Example 3 If the matrices A and B are

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 4 & 5 \\ -1 & 5 & -7 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 1+3 & 2+4 & 3+5 \\ 0+(-1) & -1+5 & 2+(-7) \end{pmatrix} = \begin{pmatrix} 4 & 6 & 8 \\ -1 & 4 & -5 \end{pmatrix}.$$

Similarly

$$A - B = \begin{pmatrix} 1-3 & 2-4 & 3-5 \\ 0-(-1) & -1-5 & 2-(-7) \end{pmatrix} = \begin{pmatrix} -2 & -2 & -2 \\ 1 & -6 & 9 \end{pmatrix}.$$

EXERCISE 1. The matrices X, Y, Z are given below.

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

Find the following matrices. (Click on the green letters for solutions.)

- (a) $X + Y$, (b) $X - Y$, (c) $Y + Z$,
(d) $(X + Y) + Z$, (e) $X + (Y + Z)$, (f) $X - (Y - Z)$.

Parts (d) and (e) above provide an example of the general rule that

$$(X + Y) + Z = X + (Y + Z).$$

NB. All of the *rules for brackets* in the addition of matrices are exactly the same as the corresponding rule for numbers.

Quiz From exercise 1, what is the 23 element of $X - (Y - Z)$?

- (a) -6 , (b) -2 , (c) 6 , (d) 10 .

If a matrix X is multiplied by a number λ , the resulting matrix, λX , is X with every element multiplied by λ . This is called *scalar multiplication*. (*Multiplying matrices* is the subject of a separate package.)

Example 4 If $\lambda = 2$ and

$$X = \begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix} \quad \text{then} \quad \lambda X = 2 \begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 10 & 4 \end{pmatrix}.$$

The *rules for brackets and scalar multiplication* of matrices are the same as for multiplication and addition of numbers.

EXERCISE 2. The matrices X, Y, Z are given below.

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \quad Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

Write out the following matrices. (Click on the **green** letters for the solutions.)

$$\begin{array}{lll} \text{(a)} \ 2(X + Y), & \text{(b)} \ 2X + 2Y, & \text{(c)} \ 2Y + Z, \\ \text{(d)} \ 2(X + Y) + Z, & \text{(e)} \ 2X - 3(Y - Z), & \text{(f)} \ 2X - 3Y + 3Z. \end{array}$$

3. The Transpose of a Matrix

The transpose of a matrix A , written A^T , is the matrix obtained by writing the rows of A as the columns of A^T .

Example 5

$$\text{If } A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 0 & -7 \end{pmatrix} \text{ then } A^T = \begin{pmatrix} -1 & 4 \\ 2 & 0 \\ 3 & -7 \end{pmatrix}.$$

EXERCISE 3. Write down the transpose of each of the following matrices. (Click on the green letters for the solutions.)

$$\text{(a) } \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 5 & -4 \end{pmatrix}, \quad \text{(b) } \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 6 & 0 & 5 \end{pmatrix}, \quad \text{(c) } (1 \ 1 \ 1),$$

Quiz If A is a $m \times n$ matrix and $A = A^T$, which of the following must *always* be true?

$$\text{(a) } m \text{ and } n \text{ may be different,} \quad \text{(b) } m = n, \quad \text{(c) } A^T \text{ is } m \times n.$$

4. Row (and Column) Operations

There are three basic *row operations* which may be performed on matrices. These are:

- (1) Multiplication of any row by a **non-zero** number.
- (2) Interchange of two rows.
- (3) Addition of a multiple of one row to another.

The following notation will be used for these operations.

- (1) $R_3 \rightarrow 4R_3$ means **row 3** is changed to $4 \times$ **row 3**.
- (2) $R_1 \leftrightarrow R_2$ means **row 1** is interchanged with **row 2**.
- (3) $R_2 \rightarrow R_2 + 3R_3$ means **row 2** has $3 \times$ **row 3** added to it.

Definition If a matrix B is obtained from A after a *sequence* of these row operations, then they are said to be *row equivalent*.

In a similar manner, *column operations* may be performed on a matrix to give a matrix which is *column equivalent* to it. Column operations performed on A are row operations performed on A^T .

Example 6

Perform the following row operations on the adjacent matrix X .

(1) $R_1 \leftrightarrow R_2$. (2) $R_2 \rightarrow -2R_2$.

(3) $R_1 \rightarrow R_1 + 2R_2$.

$$X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}$$

Solution

$$(1) \quad X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 3 & 2 \\ 1 & 2 & 0 \\ 4 & 5 & 3 \end{pmatrix}.$$

$$(2) \quad X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_2 \rightarrow -2R_2} \begin{pmatrix} 1 & 2 & 0 \\ -2 & 6 & 4 \\ 4 & 5 & 3 \end{pmatrix}.$$

$$(3) \quad X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + 2R_2} \begin{pmatrix} -1 & 8 & 4 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}.$$

Example 7

Perform the following *sequence* of row operations on the adjacent matrix X .

$R_2 \leftrightarrow R_3$, followed by $R_2 \rightarrow -2R_2$, followed by $R_1 \rightarrow R_1 + 2R_3$.

$$X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}$$

Solution In this case, the first operation is performed on X to obtain X_1 and then the second operation is performed on X_1 , to obtain a matrix X_2 say, etc.

$$\begin{array}{ccc}
 X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} & \xrightarrow{R_2 \leftrightarrow R_3} & \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 3 \\ -1 & 3 & 2 \end{pmatrix} \\
 \\
 \xrightarrow{R_2 \rightarrow -2R_2} \begin{pmatrix} 1 & 2 & 0 \\ -8 & -10 & -6 \\ -1 & 3 & 2 \end{pmatrix} & \xrightarrow{R_1 \rightarrow R_1 + 2R_3} & \begin{pmatrix} -1 & 8 & 4 \\ -8 & -10 & -6 \\ -1 & 3 & 2 \end{pmatrix}
 \end{array}$$

The final matrix is said to be *row equivalent* to X .

EXERCISE 4.

Perform the following *sequence* of row operations on the matrix Y . The resulting matrix, Y' , is said to be *upper triangular*. (Click on the *green* letters for the solutions.)

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}$$

- (a) $R_1 \rightarrow \frac{1}{2}R_1$, (b) $R_2 \rightarrow R_2 - R_1$, (c) $R_3 \rightarrow R_3 - 2R_1$.

EXERCISE 5.

Perform the following *sequence* of row operations on the matrix Z . The resulting matrix, Z' , is the *3 by 3 identity* matrix. (Click on the *green* letters for the solutions.)

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

- (a) $R_2 \rightarrow R_2 - 2R_3$, (b) $R_1 \rightarrow R_1 - R_3$, (c) $R_1 \rightarrow R_1 - 3R_2$.

5. Quiz on Matrices

The following questions all refer to the matrix A .

$$A = \begin{pmatrix} -2 & 4 & 1 \\ 3 & 5 & 7 \\ 0 & 1 & -1 \end{pmatrix}$$

Begin Quiz

- The element a_{32} is
(a) 3, (b) 1, (c) 7, (d) 4.
- If $X = A^T$ then x_{32} is
(a) 3, (b) 1, (c) 7, (d) 4.
- Perform the following *sequence* of row operations on the matrix A : $R_1 \rightarrow R_1 + R_2$ followed by $R_1 \rightarrow R_1 + 8R_3$. The 1×3 element of the resulting matrix is
(a) 3, (b) 9, (c) 0, (d) 5.

End Quiz

Solutions to Exercises

Exercise 1(a) If the matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix},$$

then

$$X+Y = \begin{pmatrix} 0+(-3) & -1+(-2) & 3+1 \\ 2+(-1) & -1+1 & 4+(-2) \end{pmatrix} = \begin{pmatrix} -3 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$

Click on the green square to return



Exercise 1(b) If the matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix},$$

then

$$X - Y = \begin{pmatrix} 0 - (-3) & -1 - (-2) & 3 - 1 \\ 2 - (-1) & -1 - 1 & 4 - (-2) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 3 & -2 & 6 \end{pmatrix}.$$

Click on the green square to return



Exercise 1(c) If the matrices Y and Z are

$$Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \quad Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

then

$$Y + Z = \begin{pmatrix} -3 + 3 & -2 + (-3) & 1 + 1 \\ -1 + 1 & 1 + 2 & -2 + 4 \end{pmatrix} = \begin{pmatrix} 0 & -5 & 2 \\ 0 & 3 & 2 \end{pmatrix}.$$

Click on the green square to return



Exercise 1(d) The three matrices X , Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

From part (a),

$$X + Y = \begin{pmatrix} -3 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$

Now add the resulting matrix $(X + Y)$ to the matrix Z .

$$(X + Y) + Z = \begin{pmatrix} -3 + 3 & -3 + (-3) & 4 + 1 \\ 1 + 1 & 0 + 2 & 2 + 4 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 5 \\ 2 & 2 & 6 \end{pmatrix}.$$

Thus

$$(X + Y) + Z = \begin{pmatrix} 0 & -6 & 5 \\ 2 & 2 & 6 \end{pmatrix}.$$

Click on the green square to return



Exercise 1(e) Given the same three matrices X , Y and Z

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

sum up them in the order, $X + (Y + Z)$. From part (c),

$$Y + Z = \begin{pmatrix} 0 & -5 & 2 \\ 0 & 3 & 2 \end{pmatrix}.$$

Adding the matrix X to the matrix $(Y + Z)$ yields

$$X + (Y + Z) = \begin{pmatrix} 0+0 & -5+(-1) & 3+2 \\ 2+0 & -1+3 & 4+2 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 5 \\ 2 & 2 & 6 \end{pmatrix}.$$

Comparing this with the previous exercise for $(X + Y) + Z$ it is seen that

$$(X + Y) + Z = X + (Y + Z).$$

Click on the green square to return



Exercise 1(f) Consider three matrices X , Y and Z

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

To find the matrix $X - (Y - Z)$, first calculate $(Y - Z)$.

$$Y - Z = \begin{pmatrix} -3 - 3 & -2 - (-3) & 1 - 1 \\ -1 - 1 & 1 - 2 & -2 - 4 \end{pmatrix} = \begin{pmatrix} -6 & 1 & 0 \\ -2 & -1 & -6 \end{pmatrix}.$$

Now subtract the resulting matrix $(Y - Z)$ from the matrix X

$$X - (Y - Z) = \begin{pmatrix} 0 - (-6) & -1 - 1 & 3 - 0 \\ 2 - (-2) & -1 - (-1) & 4 - (-6) \end{pmatrix} = \begin{pmatrix} 6 & -2 & 3 \\ 4 & 0 & 10 \end{pmatrix}$$

Thus

$$X - (Y - Z) = \begin{pmatrix} 6 & -2 & 3 \\ 4 & 0 & 10 \end{pmatrix}.$$

Click on the green square to return



Exercise 2(a) The matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}.$$

To find the matrix $2(X + Y)$, first calculate the sum $(X + Y)$.

$$X + Y = \begin{pmatrix} 0 + (-3) & -1 + (-2) & 3 + 1 \\ 2 + (-1) & -1 + 1 & 4 + (-2) \end{pmatrix} = \begin{pmatrix} -3 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

Scalar multiplication of this matrix by 2 gives

$$2 \times (X + Y) = \begin{pmatrix} 2 \times (-3) & 2 \times (-3) & 2 \times 4 \\ 2 \times 1 & 2 \times 0 & 2 \times 2 \end{pmatrix} = \begin{pmatrix} -6 & -6 & 8 \\ 2 & 0 & 4 \end{pmatrix}.$$

Click on the green square to return



Exercise 2(b) If the matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}.$$

then $2X$ and $2Y$ are

$$2X = \begin{pmatrix} 0 & -2 & 6 \\ 4 & -2 & 8 \end{pmatrix}, \quad 2Y = \begin{pmatrix} -6 & -4 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

The sum of these matrices is

$$2X+2Y = \begin{pmatrix} 0+(-6) & -2+(-4) & 6+2 \\ 4+(-2) & -2+2 & 8+(-4) \end{pmatrix} = \begin{pmatrix} -6 & -6 & 8 \\ 2 & 0 & 4 \end{pmatrix}.$$

Comparing this result with the matrix $2(X+Y)$ from part (a) shows that

$$2(X+Y) = 2X+2Y.$$

Click on the green square to return



Exercise 2(c) To find the matrix $2Y + Z$, where Y and Z are given by

$$Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \quad Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

first calculate $2Y$.

$$2Y = \begin{pmatrix} 2 \times (-3) & 2 \times (-2) & 2 \times 1 \\ 2 \times (-1) & 2 \times 1 & 2 \times (-2) \end{pmatrix} = \begin{pmatrix} -6 & -4 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

Now add this matrix to the matrix Z to obtain

$$2Y + Z = \begin{pmatrix} -6 + 3 & -4 + (-3) & 2 + 1 \\ -2 + 1 & 2 + 2 & -4 + 4 \end{pmatrix} = \begin{pmatrix} -3 & -7 & 3 \\ -1 & 4 & 0 \end{pmatrix}.$$

Click on the green square to return



Exercise 2(d) To calculate the matrix $2(X + Y) + Z$, where the matrices X , Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

first note that the matrix $2(X + Y)$ was found in part (a), i.e.

$$2(X + Y) = \begin{pmatrix} -6 & -6 & 8 \\ 2 & 0 & 4 \end{pmatrix}.$$

Now add this matrix to the matrix Z to obtain

$$2(X+Y)+Z = \begin{pmatrix} -6+3 & -6+(-3) & 8+1 \\ 2+1 & 0+2 & 4+4 \end{pmatrix} = \begin{pmatrix} -3 & -9 & 9 \\ 3 & 2 & 8 \end{pmatrix}.$$

Click on the green square to return



Exercise 2(e) The matrices X , Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \quad Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

To obtain $2X - 3(Y - Z)$, first calculate $2X$ and $3(Y - Z)$ separately.

$$2X = \begin{pmatrix} 0 & -2 & 6 \\ 4 & -2 & 8 \end{pmatrix}$$

$$\begin{aligned} 3(Y - Z) &= \begin{pmatrix} 3 \times (-3 - 3) & 3 \times (-2 - (-3)) & 3 \times (1 - 1) \\ 3 \times (-1 - 1) & 3 \times (1 - 2) & 3 \times (-2 - 4) \end{pmatrix} \\ &= \begin{pmatrix} -18 & 3 & 0 \\ -6 & -3 & -18 \end{pmatrix} \end{aligned}$$

Now subtract $3(Y - Z)$ from $2X$ to obtain

$$2X - 3(Y - Z) = \begin{pmatrix} 18 & -5 & 6 \\ 10 & 1 & 26 \end{pmatrix}.$$

Click on the green square to return



Exercise 2(f) The matrices X , Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

To find $C = 2X - 3Y + 3Z$, first find $2X$, $3Y$ and $3Z$:

$$2X = \begin{pmatrix} 0 & -2 & 6 \\ 4 & -2 & 8 \end{pmatrix}, 3Y = \begin{pmatrix} -9 & -6 & 3 \\ -3 & 3 & -6 \end{pmatrix}, 3Z = \begin{pmatrix} 9 & -9 & 3 \\ 3 & 6 & 12 \end{pmatrix}.$$

The matrix C is now

$$\begin{aligned} C &= \begin{pmatrix} 0 - (-9) + 9 & -2 - (-6) + (-9) & 6 - 3 + 3 \\ 4 - (-3) + 3 & -2 - 3 + 6 & 8 - (-6) + 12 \end{pmatrix} \\ &= \begin{pmatrix} 18 & -5 & 6 \\ 10 & 1 & 26 \end{pmatrix}. \end{aligned}$$

Comparing this with the result of [Exercise 2\(e\)](#) it is seen that

$$2X - 3(Y - Z) = 2X - 3Y + 3Z.$$

[Click on the green square to return](#)



Exercise 3(a)

If the 3×2 matrix A is

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 5 & -4 \end{pmatrix},$$

then its **transpose** is the 2×3 matrix, A^T , whose **columns** are the **rows** of the matrix A :

$$A^T = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 4 & -4 \end{pmatrix}.$$

Click on the green square to return



Exercise 3(b)

If the matrix B is the 3×3 matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 6 & 0 & 5 \end{pmatrix},$$

then its **transpose** matrix, B^T , is also a 3×3 matrix but the **rows** of B are the **columns** of B^T . Thus

$$B^T = \begin{pmatrix} 1 & 3 & 6 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{pmatrix}.$$

Click on the green square to return



Exercise 3(c)

If a matrix X has a single row

$$X = (1 \quad 1 \quad 1) ,$$

then its transpose matrix, X^T , is a matrix with a single column

$$X^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

The transpose of a row matrix is a column matrix.

Matrices such as X are sometimes referred to as row vectors and matrices such as X^T are sometimes called column vectors.

Click on the green square to return



Exercise 4(a) If Y is the matrix

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix},$$

then performing the **row operation** $R_1 \rightarrow \frac{1}{2}R_1$, we obtain the matrix

$$Y_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}.$$

Click on the green square to return



Exercise 4(b) After the first row operation $R_1 \rightarrow \frac{1}{2}R_1$ the matrix Y transformed into the matrix Y_1

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{2}R_1} Y_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}.$$

Now applying the second row operation $R_2 \rightarrow R_2 - R_1$ to Y_1 gives

$$Y_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 7 \end{pmatrix}.$$

Click on the green square to return



Exercise 4(c) After performing the two subsequent **row operations** the matrix Y was transformed into the matrix Y_2 :

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{2}R_1} Y_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}$$
$$\xrightarrow{R_2 \rightarrow R_2 - R_1} Y_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 7 \end{pmatrix}.$$

Now applying the **row operation** $R_3 \rightarrow R_3 - 2R_1$ to Y_2 we arrive at the **row equivalent** matrix Y' where

$$Y' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

with entries below the diagonal which are all zero. Matrices of this type are called the *upper triangular matrices*.

Click on the green square to return



Exercise 5(a) Performing the row operation $R_2 \rightarrow R_2 - 2R_3$ on Z gives

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_3} Z_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Click on the green square to return



Exercise 5(b) After the first row operation $R_2 \rightarrow R_2 - 2R_3$ the matrix Z was transformed into the matrix Z_1

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_3} Z_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Performing now the row operation $R_1 \rightarrow R_1 - R_3$, we have

$$Z_2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Click on the green square to return



Exercise 5(c) As a result of the two subsequent **row operations** the matrix Z was transformed into the matrix Z_2 :

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_3} Z_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{R_1 \rightarrow R_1 - R_3} Z_2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now applying the **row operation** $R_1 \rightarrow R_1 - 3R_2$ to Z_2 gives

$$Z' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The resulting matrix is the 3×3 **identity** matrix: the elements along its diagonal are $z'_{ii} = 1$ and all other elements are zero.

Click on the green square to return



Solutions to Quizzes

Solution to Quiz: The matrix is

$$C = \begin{pmatrix} 5 & -3 & 7 \\ 7 & 0 & -7 \\ 0 & 25 & 0 \end{pmatrix}$$

and -7 is in the **second row** and **third column**, so it is c_{23} , i.e. the 2×3 element.

End Quiz

Solution to Quiz:

From exercise 1(f), the matrix $C = X - (Y - Z)$ is

$$C = X - (Y - Z) = \begin{pmatrix} 6 & -2 & 3 \\ 4 & 0 & 10 \end{pmatrix}.$$

The 23 element of this matrix is the element in the second row and third column, therefore

$$c_{23} = 10.$$

End Quiz

Solution to Quiz: The matrix equality

$$A = B$$

means that

$$\begin{aligned} \text{number of rows of } A &= \text{number of rows of } B \\ \text{number of columns of } A &= \text{number of columns of } B, \end{aligned}$$

and every element of one matrix is equal to the corresponding element of the other matrix.

If A is a matrix of order $m \times n$, then the **transpose** matrix has n -rows and m -columns, i.e. A^T is matrix of order $n \times m$. Therefore the equality

$$A = A^T$$

requires at least that the number of rows is the same as the number of columns:

$$m = n.$$

(N.B. A matrix which has the same number of rows as columns is called a **square matrix**.)

End Quiz