PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

Maxima and Minima

R Horan & M Lavelle

The aim of this document is to provide a short, self assessment programme for students who wish to be able to use differentiation to find maxima and mininima of functions.

Last Revision Date: May 5, 2005 Version 1.0

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1. Rules of Differentiation

Throughout this package the following rules of differentiation will be assumed. (In the table of derivatives below, a is an arbitrary, non-zero constant.)

y	ax^n	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\ln(ax)$
$\dfrac{dy}{dx}$	nax^{n-1}	$a\cos(ax)$	$-a\sin(ax)$	ae^{ax}	$\dfrac{1}{x}$

If a is any constant and u, v are two functions of x, then

$$\begin{aligned} \frac{d}{dx}(u+v) &= \frac{du}{dx} + \frac{dv}{dx} \\ \frac{d}{dx}(au) &= a\frac{du}{dx} \end{aligned}$$

The *stationary points* of a function are those points where the gradient of the tangent (the derivative of the function) is zero.

Example 1 Find the stationary points of the functions

$$\text{(a) } f(x) = 3x^2 + 2x - 9\,, \quad \text{(b) } f(x) = x^3 - 6x^2 + 9x - 2\,.$$

Solution

(a) If $y = 3x^2 + 2x - 9$ then $\frac{dy}{dx} = 3 \times 2x^{2-1} + 2 = 6x + 2$.

The stationary points are found by solving the equation

$$\frac{dy}{dx} = 6x + 2 = 0\,.$$

In this case there is only one solution, $x = -1/3$. Substituting this into the equation for $f(x)$, the corresponding y value is then

$$\begin{aligned} y &= 3(-1/3)^2 + 2(-1/3) - 9 \\ &= -28/3\,. \end{aligned}$$

The stationary point is thus the point with coordinates $(-1/3, -28/3)$.

(b) If $y = f(x) = x^3 - 6x^2 + 9x - 2$, then $\frac{dy}{dx} = 3x^2 - 12x + 9$.

The stationary points are found by solving $3x^2 - 12x + 9 = 0$. This is a quadratic equation (see the package on **quadratic equations**) and may be solved by factorising.

$$\begin{aligned} 0 = 3x^2 - 12x + 9 &= 3(x^2 - 4x + 3) \\ &= 3(x-3)(x-1)\,. \end{aligned}$$

There are two solutions in this case, $x = 3$ and $x = 1$. If these are substituted into the function, the two stationary points will be found to be $(3, -2)$ and $(1, 2)$.

Exercise 1. Find the *stationary points* of the following functions. (Click on the green letters for solutions.)

(a) $y = 16 - 6u - u^2$,

(b) $y = 3x^2 - 4x + 7$,

(c) $y = \frac{1}{3}x^3 - x^2 - 3x + 2$,

(d) $y = x^3 - 6x^2 - 15x + 16$.

2. Derivatives of order 2 (and higher)

In the package on **introductory differentiation** the height, $h(t)$, in metres, of a ball thrown vertically at $10\,\mathrm{ms}^{-1}$, was given by

$$h(t) = 10t - 5t^2\,.$$

The *velocity* of the ball, $v\,\mathrm{ms}^{-1}$, after t seconds, was given by

$$v(t) = \frac{dh}{dt} = 10 - 10t\,.$$

The rate of change of velocity with time, which is the *acceleration*, is then given by $a(t)$, where

$$a(t) = \frac{dv}{dt} = -10\,\mathrm{ms}^{-1}\,.$$

Since the acceleration was derived from $h(t)$ by two successive differentiations, the resulting function, which in this case is $a(t)$, is called the *second derivative* of $h(t)$ with respect to t. In symbols

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dh}{dt}\right) = \frac{d^2h}{dt^2}\,.$$

(Note the position of the superscripts)

Example 2

Find the *first*, *second* and *third* derivatives of the function

$$y = 3x^6 - 2x + 1\,.$$

Solution

The *first* derivative is given by

$$\frac{dy}{dx} = 18x^5 - 2\,.$$

The *second* derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 90x^4\,.$$

The *third* derivative is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 360x^3\,.$$

Exercise 2. Find the *first*, *second* and *third* derivatives of the following functions. (Click on the green letters for solutions.)

(a) $y = x^3 + \mathrm{e}^{2x}$, (b) $y = \sin(x)$.

3. Maxima and Minima

The diagram below shows part of a function $y = f(x)$.

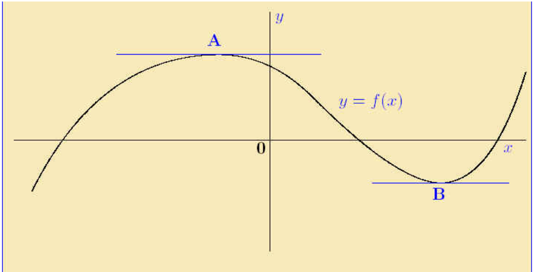

Diagram 1

The point A is a *local maximum* and the point B is a *local minimum*. At each of these points the tangent to the curve is parallel to the x-axis so the derivative of the function is zero. Both of these points are therefore *stationary points* of the function. The term *local* is used since these points are the maximum and minimum in this particular region. There may be others outside this region.

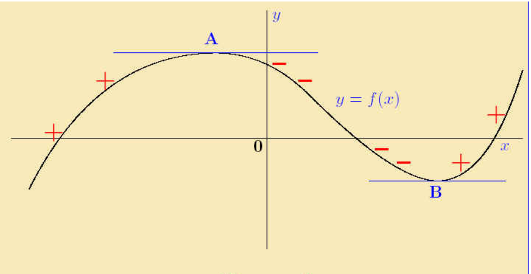

Diagram 2

Further observations on the gradients of tangents to the curve are:

to the *left* of A the gradients are positive (+)
between A and B the gradients are negative (−)
to the *right* of B the gradients are positive (+)

About the *local maximum* point A the gradient changes from positive, to zero, to negative. The gradient is therefore *decreasing*.
About the *local minimum* point B the gradient changes from negative, to zero, to positive. The gradient is therefore *increasing*.

The rate of change of a function is measured by its derivative. When the derivative is *positive*, the function is *increasing*, when the derivative is *negative*, the function is *decreasing*. Thus the rate of change of the *gradient* is measured by *its derivative*, which is the *second derivative* of the original function. In mathematical notation this is as follows.

At the point (a, b)

if	$\frac{dy}{dx}$	$=$	0	if	$\frac{dy}{dx}$	$=$	0
and	$\frac{d^2y}{dx^2}$	$<$	0	and	$\frac{d^2y}{dx^2}$	$>$	0
then the point	(a, b)	is a		then the point	(a, b)	is a	
local maximum.				**local minimum.**			

Example 3

Find the stationary point of the function $y = x^2 - 2x + 3$ and hence determine the nature of this point.

Solution

If $y = x^2 - 2x + 3$ then $\dfrac{dy}{dx} = 2x - 2$ and $\dfrac{d^2y}{dx^2} = 2$.

Now $\dfrac{dy}{dx} = 2x - 2 = 0$ when $x = 1$. The function has only one stationary point when $x = 1$ (and $y = 2$). Since $\dfrac{d^2y}{dx^2} = 2 > 0$ for all values of x, this stationary point is a *local minimum*. Thus the function $y = x^2 - 2x + 3$ has a *local minimum* at the point $(1, 2)$.

The graph shows the function $y = x^2 - 2x + 3$ with the local minimum point at (1,2) clearly visible.

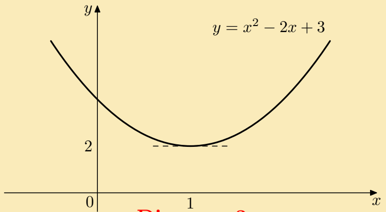

Diagram 3

Exercise 3. Determine the nature of the *stationary points* of the following functions. (Click on the green letters for solutions.)

(a) $y = 16 - 6u - u^2$, (b) $y = 3x^2 - 4x + 7$,

Quiz Which of the following points is a local maximum of the function $y = 2x^3 - 15x^2 - 36x + 6$?

(a) $(-3, 15)$, (b) $(-6, 25)$, (c) $(-2, 25)$, (d) $(-1, 25)$.

Example 4

Find the stationary points of the function $y = 2x^3 - 9x^2 + 12x - 3$ and determine their nature.

Solution

$$\begin{aligned} \text{If} \quad y &= 2x^3 - 9x^2 + 12x - 3 \\ \text{then} \quad \frac{dy}{dx} &= 6x^2 - 18x + 12 \\ \text{and} \quad \frac{d^2y}{dx^2} &= 12x - 18\,. \end{aligned}$$

The stationary points are found by solving

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 0\,.$$

Now $6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$ so the solution is found by solving $x^2 - 3x + 2 = 0$, i.e. $x = 1$ and $x = 2$. (See the package on **quadratics**.) **Note** that when $x = 1,\ y = 2$ and when $x = 2,\ y = 1$.

When $x = 1,\ \dfrac{d^2y}{dx^2} = 12(1) - 18 = -6 < 0$, so $(1, 2)$ is a *local maximum*.

When $x = 2,\ \dfrac{d^2y}{dx^2} = 12(2) - 18 = +6 > 0$, so $(2, 1)$ is a *local minimum*.

The graph shows the function $y = 2x^3 - 9x^2 + 12x - 3$ with the *local maximum* point at (1,2) and the *local minimum* point at (2,1) clearly visible.

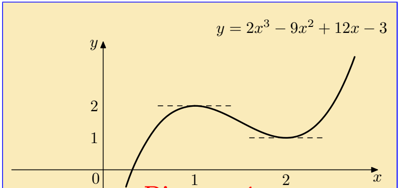

Diagram 4

EXERCISE 4. Determine the nature of the *stationary points* of the following functions. (Click on the green letters for solutions.)

(a) $y = \frac{1}{3}x^3 - x^2 - 3x + 2$, (b) $y = \frac{1}{3}s^2 - 12s + 32$,

(c) $y = x^3 - 6x^2 - 15x + 16$, (d) $y = x^3 - 12x + 12$.

Quiz Referring to the function $y = x + 1 + \dfrac{1}{x}$, which of the following statements is true?

(a) $(1, 3)$ is a *local maximum*, (b) $(1, -3)$ is a *local minimum*,

(c) $(-1, 1)$ is a *local minimum*, (d) $(-1, -1)$ is a *local maximum*.

Quiz Referring to the function $y = \dfrac{x^2}{2} - \cos x$, which of the following statements is true?

(a) $(\frac{\pi}{2}, \frac{\pi^2}{8})$ is a *local minimum*, (b) $(\frac{\pi}{2}, \frac{\pi^2}{8})$ is a *local maximum*,

(c) $(0, -1)$ is a *local minimum*, (d) $(0, -1)$ is a *local maximum*.

4. Quiz on Max and Min

Begin Quiz In each of the following cases, choose the correct option.

1. If $y = 3x^2 + 6x + 2$ then
(a) $(1, 11)$ is a *local minimum*, (b) $(-1, -1)$ is a *local minimum*,
(c) $(1, 11)$ is a *local maximum*, (d) $(-1, -1)$ is a *local maximum*.

2. The function $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 4$ has a
(a) *local minimum* when $x = 2$, (b) *local maximum* when $x = -2$,
(c) *local maximum* when $x = 1$, (d) *local minimum* when $x = -1$.

3. The function $y = \sin x + \cos x$ has a
(a) *local maximum* when $x = \frac{\pi}{4}$, (b) *local minimum* when $x = \frac{\pi}{4}$,
(c) *local maximum* when $x = \frac{\pi}{2}$, (d) *local minimum* when $x = \frac{\pi}{2}$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) If $y = f(u) = 16 - 6u - u^2$, then its derivative is

$$\frac{dy}{du} = -6 - 2u\,.$$

The stationary points are found by solving the linear equation

$$-6 - 2u = 0\,.$$

In this case there is only one solution $u = -\frac{6}{2} = -3$. Substituting this into the expression for the function $f(u) = 16 - 6u - u^2$, we find the corresponding y value

$$y = 16 - 6 \times (-3) - (-3)^2 = 16 + 18 - 9 = 25\,.$$

The stationary point is thus the point with coordinates $(-3, 25)$.
Click on the green square to return

□

Exercise 1(b) If $y = f(x) = 3x^2 - 4x + 7$, then its derivative is

$$\frac{dy}{dx} = 6x - 4\,.$$

The stationary points are found by solving the linear equation

$$6x - 4 = 0\,.$$

In this case there is only one solution $x = \frac{4}{6} = \frac{2}{3}$. Substituting this into the expression for the function $f(x) = 3x^2 - 4x + 7$, we find the corresponding y value

$$y = 3 \times \left(\frac{2}{3}\right)^2 - 4 \times \frac{2}{3} + 7 = \frac{4}{3} - \frac{8}{3} + 7 = \frac{17}{3}\,.$$

Therefore the function $y = 3x^2 - 4x + 7$ has only one stationary point $(\frac{2}{3}, \frac{17}{3})$.

Click on the green square to return

□

Exercise 1(c) If $y = f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$, then $\frac{dy}{dx} = x^2 - 2x - 3$. The stationary points are found by solving the quadratic equation $x^2 - 2x - 3 = 0$. Factorisation of this equation gives

$$0 = x^2 - 2x - 3 = (x+1)(x-3)\,.$$

yielding two solutions, $x = -1$ and $x = 3$. Substituting $x = -1$ into the function $f(x)$, the corresponding y value is

$$y = \frac{1}{3}(-1)^3 - (-1)^2 - 3(-1) + 2 = -\frac{1}{3} - 1 + 3 + 2 = \frac{11}{3}\,.$$

Similarly substituting $x = 3$

$$y = \frac{1}{3}(3)^3 - (3)^2 - 3(3) + 2 = 9 - 9 - 9 + 2 = -7\,.$$

The two stationary point are thus $(-1, \frac{11}{3})$ and $(3, -7)$.

Click on the green square to return

Exercise 1(d) If $y = f(x) = x^3 - 6x^2 - 15x + 16$, then

$$\frac{dy}{dx} = 3x^2 - 12x - 15\,.$$

The stationary points are found by solving the quadratic equation $3x^2 - 12x - 15 = 0$. Factorising it

$$0 = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x+1)(x-5)$$

we obtain two solutions, $x = -1$ and $x = 5$. Substituting $x = -1$ into the function $f(x)$, the corresponding y value is

$$y = (-1)^3 - 6 \times (-1)^2 - 15 \times (-1) + 16 = -1 - 6 + 15 + 16 = 24\,.$$

Similarly substituting $x = 5$

$$y = (5)^3 - 6 \times (5)^2 - 15 \times (5) + 16 = 125 - 6 \times 25 - 15 \times 5 + 16 = -84\,.$$

Two stationary point are thus $(-1, 24)$ and $(5, -84)$.

Click on the green square to return

□

Exercise 2(a) If $y = x^3 + \mathrm{e}^{2x}$ then the *first* derivative is given by

$$\frac{dy}{dx} = 3 \times x^{(3-1)} + 2 \times \mathrm{e}^{2x} = 3x^2 + 2\mathrm{e}^{2x}\,.$$

The *second* derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2 \times 3x^{(2-1)} + 2 \times 2\mathrm{e}^{2x} = 6x + 4\mathrm{e}^{2x}\,.$$

The *third* derivative is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 6 + 8\mathrm{e}^{2x}\,.$$

Click on the green square to return

Exercise 2(b) If $y = \sin(x)$ then the *first* derivative is given by

$$\frac{dy}{dx} = \cos(x)\,.$$

The *second* derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\sin(x),.$$

The *third* derivative is given by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = -\cos(x)\,.$$

Click on the green square to return

□

Exercise 3(a) If $y = 16 - 6u - u^2$ then

$$\frac{dy}{du} = -6 - 2u \quad \text{and} \quad \frac{d^2y}{du^2} = -2\,.$$

There is one stationary point when $u = -3$ which is obtained from the equation

$$\frac{dy}{du} = -6 - 2u = 0\,.$$

At this point $y = 16 - 6 \times (-3) - (-3)^2 = 25\,.$ Since $\frac{d^2y}{du^2} = -2 < 0$ for all values of x, this stationary point is a *local maximum*. Thus the function $y = 16 - 6u - u^2$ has a *local maximum* at the point $(-3, 25)$.

Click on the green square to return

Exercise 3(b) If $y = 3x^2 - 4x + 7$ then

$$\frac{dy}{dx} = 6x - 4 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6\,.$$

There is one stationary point $x = \dfrac{2}{3}$ which is the solution to the equation

$$\frac{dy}{dx} = 6x - 4 = 0\,.$$

At this point $y = 3 \times \left(\dfrac{2}{3}\right)^2 - 4 \times \left(\dfrac{2}{3}\right) + 7 = \frac{17}{3}$. Since $\dfrac{d^2y}{dx^2} = 6 > 0$ for all values of x, this stationary point is a *local minimum*. Thus the function $y = 3x^2 - 4x + 7$ has a *local minimum* at the point $(\frac{2}{3}, \frac{17}{3})$.

Click on the green square to return

□

Exercise 4(a) If $y = f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$ then

$$\frac{dy}{dx} = x^2 - 2x - 3 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2x - 2\,.$$

There are two stationary points at $x = -1$ and $x = 3$ which are the two solutions to the quadratic equation

$$\frac{dy}{dx} = x^2 - 2x - 3 = (x+1)(x-3) = 0\,.$$

When $x = -1$, $y = \frac{11}{3}$ and its second derivative has the value $\frac{d^2y}{dx^2} = 2\times(-1)-2 = -4 < 0$. The point $(-1, \frac{11}{3})$ is a *local maximum*.
At the second point $x = 3$ and $y = -7$. The second derivative has the value $\frac{d^2y}{dx^2} = 2 \times (3) - 2 = 4 > 0$ and the point $(3, -7)$ is a *local minimum*.

Click on the green square to return

Exercise 4(b) If $y = f(s) = \frac{1}{3}s^2 - 12s + 32$ then

$$\frac{dy}{ds} = \frac{2}{3}s - 12 \quad \text{and} \quad \frac{d^2y}{ds^2} = \frac{2}{3}\,.$$

There is only one stationary point, when $s = 18$, which is obtained from the equation

$$\frac{dy}{ds} = \frac{2}{3}s - 12 = \frac{2}{3}(s - 18) = 0\,.$$

When $s = 18$ the function $f(s)$ takes the value $y = -76$. Since its second derivative $\frac{d^2y}{ds^2} = \frac{2}{3}$ is positive for all values of s, the stationary point $(18, -76)$ is a *local minimum*.

Click on the green square to return

□

Exercise 4(c) If $y = f(x) = x^3 - 6x^2 - 15x + 16$ then

$$\frac{dy}{dx} = 3x^2 - 12x - 15 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x - 12\,.$$

There are two stationary points $x = -1$ and $x = 5$ which are obtained from solving the quadratic equation

$$\frac{dy}{dx} = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x+1)(x-5) = 0\,.$$

When $x = -1$ the function value is $y = 24$ and its second derivative is $\frac{d^2y}{dx^2} = 6 \times (-1) - 12 = -18 < 0$. The point $(-1, 24)$ is a *local maximum*.

At the second point, $x = 5$, the function value is $y = -84$ and the second derivative is $\frac{d^2y}{dx^2} = 6 \times (5) - 12 = 18 > 0$. The point $(5, -84)$ is a *local minimum*.

Click on the green square to return □

Exercise 4(d) If $y = f(x) = x^3 - 12x + 12$ then

$$\frac{dy}{dx} = 3x^2 - 12 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x\,.$$

There are two stationary points $x = -2$ and $x = 2$ which are obtained from the equation

$$\frac{dy}{dx} = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2) = 0\,.$$

At the point $x = -2$ the function value is $y = 28$ and its second derivative is $\frac{d^2y}{dx^2} = 6 \times (-2) = -12 < 0$. The point $(-2, 28)$ is therefore a *local maximum*.

At the second point $x = 2$ the function value is $y = -4$ and the second derivative is $\frac{d^2y}{dx^2} = 6 \times (2) = 12 > 0$. The point $(2, -4)$ is a *local minimum*.

Click on the green square to return

Solutions to Quizzes

Solution to Quiz: If $y = f(x) = 2x^3 - 15x^2 - 36x + 6$ then

$$\frac{dy}{dx} = 6x^2 - 30x - 36 \quad \text{and} \quad \frac{d^2y}{dx^2} = 12x - 30\,.$$

There are two stationary points $x = -1$ and $x = 6$ which are obtained from solving the equation

$$0 = \frac{dy}{dx} = 6x^2 - 30x - 36 = 6(x^2 - 5x - 6) = 6(x+1)(x-6)\,.$$

At the point $x = -1$ the function value is $y = 25$ and its second derivative is $\frac{d^2y}{dx^2} = 12 \times (-1) - 30 = -42 < 0$. Therefore the point $(-1, 25)$ is a *local maximum*.

For the second stationary point, at $x = 6$, the second derivative is $\frac{d^2y}{dx^2} = 12 \times (6) - 30 = 42 > 0$ and $y = -318$. The point $(6, -318)$ is therefore a *local minimum*. End Quiz

Solution to Quiz: If $y = x + 1 + \dfrac{1}{x} = x + 1 + x^{-1}$ then

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = -(-2)x^{-3} = \frac{2}{x^3}\,.$$

The stationary points are found from the equation

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{1 - x^2}{x^2} = 0\,,$$

and this is zero when $1 - x^2 = 0$. There are two solutions in this case, $x = -1$ and $x = 1$.

When $x = -1$ the function value is $y = -1$ and its second derivative is $\dfrac{d^2y}{dx^2} = \dfrac{2}{(-1)^3} = -2 < 0$. The point $(-1, -1)$ is a *local maximum*.

When $x = 1$ the function value is $y = 3$ and the second derivative is $\dfrac{d^2y}{dx^2} = \dfrac{2}{(1)^3} = 2 > 0$. The point $(1, 3)$ is a *local minimum*.

End Quiz

Solution to Quiz: If $y = \dfrac{x^2}{2} - \cos(x)$ then

$$\frac{dy}{dx} = x + \sin(x) \quad \text{and} \quad \frac{d^2y}{dx^2} = 1 + \cos(x)\,.$$

The stationary points are found from the equation

$$\frac{dy}{dx} = x + \sin(x) = 0\,.$$

Since $\sin(0) = 0$, a solution to this equation is $x = 0$. At this point the function value is

$$y = 0 - \cos(0) = 0 - 1 = -1\,.$$

Its second derivative when $x = 0$ is

$$\frac{d^2y}{dx^2} = 1 + \cos(0) = 1 + 1 = 2 > 0\,.$$

The point $(0, -1)$ is therefore a *local minimum* of the function

$$y = \frac{x^2}{2} - \cos(x)\,.$$

End Quiz