PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

Log-Log Plots

R Horan & M Lavelle

The aim of this package is to provide a short self assessment programme for students who wish to acquire an understanding of log-log plots.

Last Revision Date: June 1, 2004 Version 1.1

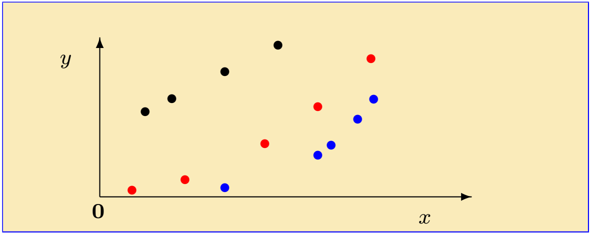
Table of Contents

1. Introduction
2. Straight Lines from Curves
3. Fitting Data
4. Final Quiz
Solutions to Exercises
Solutions to Quizzes

The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

Many quantities in science can be described by equations of the form, $y = Ax^n$. It is, though, not easy to distinguish between graphs of different power laws. Consider the data below:

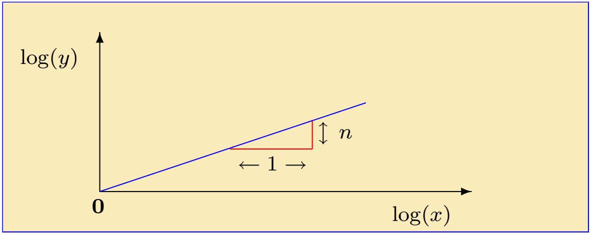


It is not easy to see that the red points lie on a quadratic ($y = Ax^2$) and that the blue data are on a quartic ($y = Ax^4$). It is, however, clear that the **black** points lie on a straight line! Results from the packages on **Logarithms** and **Straight Lines** enable us to recast the power curves as straight lines and so extract both n and A.

Example 1 Consider the equation $y = x^n$. This is a power curve, but if we take the logarithm of each side we obtain:

$$\begin{aligned}\log(y) &= \log(x^n) \\ &= n\log(x) \qquad \text{since} \quad \log(x^n) = n\log(x)\end{aligned}$$

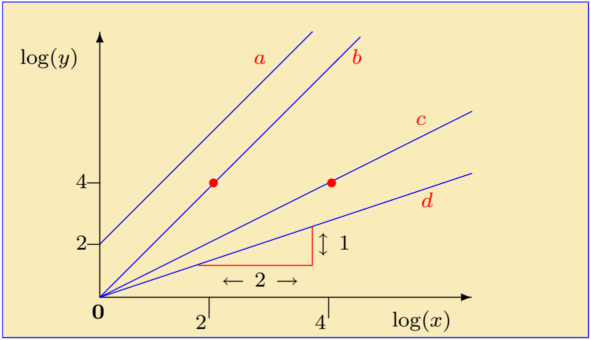
If $Y = \log(y)$ and $X = \log(x)$ then $Y = nX$. This shows the linear relationship. Plotting Y against X, i.e., $\log(y)$ against $\log(x)$, leads to a straight line as shown below.



Here n is the slope of the line. Thus:

from a log-log plot, we can directly read off the power, n.

Quiz Which of the following lines is a log-log plot of $y = x^2$?



(a) *a* (b) *b* (c) *c* (d) *d*

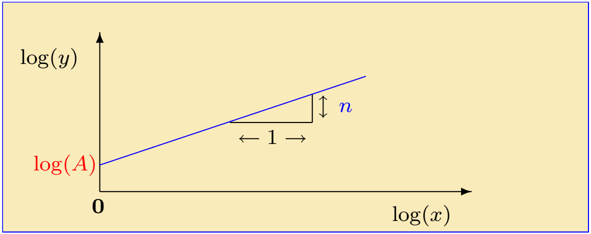
Note that the scales on the two axes are not the same.

2. Straight Lines from Curves

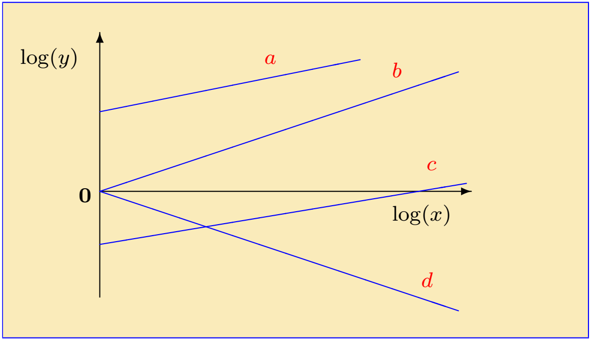
Example 2 Consider the more general equation $y = Ax^n$. Again we take the logarithm of each side:

$$\begin{aligned} \log(y) &= \log(Ax^n) \\ &= \log(A) + \log(x^n) \qquad \text{since} \quad \log(pq) = \log(p) + \log(q) \\ \therefore \log(y) &= n\log(x) + \log(A) \qquad \text{since} \quad \log(x^n) = n\log(x) \end{aligned}$$

The function $\log(y)$ is a linear function of $\log(x)$ and its graph is a straight line with gradient n which intercepts the $\log(y)$ axis at $\log(A)$.



Quiz Referring to the lines, a, b, c and d below, which of the following statements is **NOT** correct?



(a) If b corresponds to $y = x^3$, then d would describe $y = x^{-3}$.
(b) Lines a and c correspond to curves with the same power n.
(c) In the power law yielding c the coefficient A is negative.
(d) If b is from $y = x^3$, then in a the power n satisfies: $0 < n < 3$.

Exercise 1. Produce log-log plots for each of the following power curves. In each case give the gradient and the intercept on the $\log(y)$ axis. (Click on the **green** letters for the solutions).

(a) $y = x^{\frac{1}{3}}$
(b) $y = 10x^5$
(c) $y = 10x^{-2}$
(d) $y = \frac{1}{3}x^{-3}$

Quiz How does changing the base of the logarithm used (e.g., using $\ln(x)$ instead of $\log_{10}(x)$), change a log-log plot?

(a) The log-log plot is unchanged.
(b) Only the gradient changes.
(c) Only the intercept changes.
(d) Both the gradient and the intercept change.

Note that in an equation of the form $y = 5+3x^2$, taking logs directly does not help. This is because there is no rule to simplify $\log(5+3x^2)$. Instead we have to subtract the constant from each side.We then get: $y - 5 = 3x^2$, which leads to the straight line equation: $\log(y - 5) = 2\log(x) + \log(3)$.

3. Fitting Data

Suppose we want to see if some experimental data fits a power law of the form, $y = Ax^n$. We take logs of both sides and plot the points on a graph of $\log(y)$ against $\log(x)$. If they lie on a straight line (within experimental accuracy) then we conclude that y and x are related by a power law and the parameters A and n can be deduced from the graph. If the points do not lie on a straight line, then x and y are not related by an equation of this form.

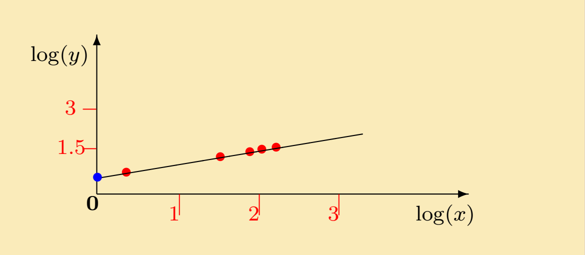
Example 3 Consider the following data:

x	2	30	70	100	150
y	4.24	16.4	25.1	30.0	36.7

To see if it obeys, $y = Ax^n$, we take logarithms of both sides. Here we use logarithms to the base 10. This gives the new table:

$\log_{10}(x)$	0.30	1.48	1.85	2	2.18
$\log_{10}(y)$	0.63	1.21	1.40	1.48	1.56

This is plotted on the next page.



It is evident that the red data points lie on a straight line. Therefore the original x and y values are related by a power law $y = Ax^n$.
To find the values of A and n, we first continue the line to the $\log_{10}(y)$ axis which it intercepts at the blue dot: $\log_{10}(A) = 0.48$. This means that $A = 10^{0.48} = 3.0$ (to 1 d.p.).
The gradient of the line is estimated using two of the points

$$n = \frac{\log(y_2) - \log(y_1)}{\log(x_2) - \log(x_1)} = \frac{1.56 - 0.63}{2.18 - 0.3} = 0.5 \quad \text{(to 1 d.p.)}$$

So the original data lies on the curve: $y = 3x^{\frac{1}{2}}$

Exercise 2. In the exercises below click on the **green** letters for the solutions.

(a) Rewrite the following expression in such a way that it gives the equation of a straight line

$$y = \sqrt{4x} + 2$$

(b) What is the difference between two power laws if, when they are plotted as a log-log graph, the gradients are the same, but the $\log(y)$ intercepts differ by $\log(3)$?

(c) Produce a log-log plot for the following data, show it obeys a power law and extract the law from the data.

x	5	15	30	50	95
y	10	90	360	1000	3610

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. The intercept and slope respectively of the log-log plot of $y = \frac{1}{2}x^2$

(a) $\frac{1}{2}$ & $\log(2)$ (b) $-\log(2)$ & 2

(c) $\log(2)$ & 2 (d) $\log(1/2)$ & $\log(2)$

2. If the $\log(y)$ axis intercept of the log-log plot of $y = Ax^n$ is negative, which of the following statements is true.

(a) $n < 0$ (b) $A = 1$ (c) $0 < A < 1$ (d) $n = -A$

3. The data below obeys a power law, $y = Ax^n$. Obtain the equation and select the correct statement.

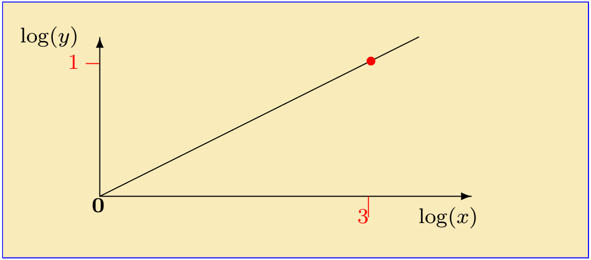
x	5	15	30	50	95
y	10	90	360	1000	3610

(a) $n = 3$ (b) $A = \frac{3}{2}$ (c) $n = 4$ (d) $A = \frac{1}{2}$

End Quiz Score: Correct

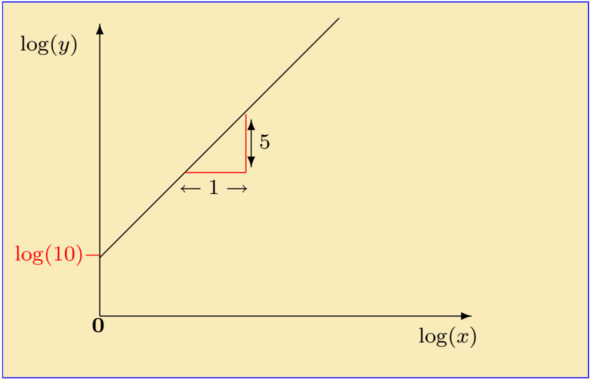
Solutions to Exercises

Exercise 1(a) For $y = x^{\frac{1}{3}}$, we get on taking logs: $\log(y) = \frac{1}{3}\log(x)$. This describes a line that passes through the origin and has slope $\frac{1}{3}$. It is sketched below:



Click on the **green** square to return □

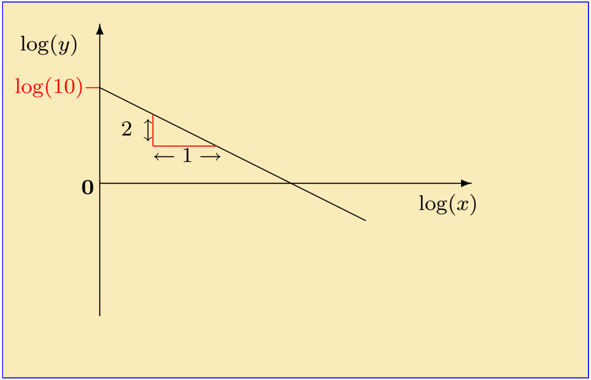
Exercise 1(b) For $y = 10x^5$, we get on taking logarithms of each side: $\log(y) = 5\log(x) + \log(10)$. This describes a line that passes through $(0, \log(10))$ and has slope 5. It is sketched below:



Click on the **green** square to return

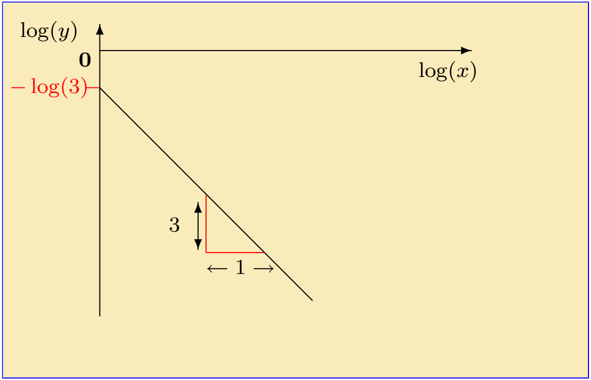
□

Exercise 1(c) The relation $y = 10x^{-2}$, can be re-expressed as $\log(y) = -2\log(x) + \log(10)$. This is sketched below.



Click on the **green** square to return

Exercise 1(d) If $y = \frac{1}{3}x^{-3}$, then $\log(y) = -3\log(x) + \log(\frac{1}{3})$. This can also be written as $\log(y) = -3\log(x) - \log(3)$. It is the equation of a line with slope -3 and intercept at $-\log(3)$. The line is sketched below.



Click on the **green** square to return □

Exercise 2(a) $y = \sqrt{4x} + 4$ can be re-expressed as follows. Subtract 4 from each side

$$\begin{aligned} y - 4 &= \sqrt{4x} \\ y - 4 &= 2\sqrt{x} \\ y - 4 &= 2x^{\frac{1}{2}} \end{aligned}$$

Taking logarithms of each side yields

$$\log(y-4) = \frac{1}{2}\log(x) + \log(2)$$

Thus plotting $\log(y-4)$ against $\log(x)$ would give a straight line with slope $\frac{1}{2}$ and intercept $\log(2)$ on the $\log(y-4)$ axis.

Click on the **green** square to return

□

Exercise 2(b) If $y = Ax^n$ then the log-log plot is the graph of the straight line

$$\log(y) = n\log(x) + \log(A)$$

So if the slope is the same the power n is the same in each case.

If the coefficients A_1 and A_2 differ by

$$\begin{aligned}\log(A_1) - \log(A_2) &= \log(3)\\ \text{then } \log\left(\frac{A_1}{A_2}\right) &= \log(3) \text{ since } \log(p/q) = \log(p) - \log(q)\end{aligned}$$

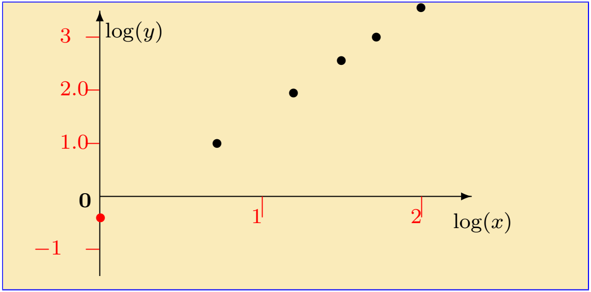
so it follows that the coefficients are related by $A_1 = 3A_2$.

Click on the **green** square to return

□

Exercise 2(c) To see if it obeys, $y = Ax^n$, we take logarithms to the base 10 of both sides. The table and graph are below:

$\log_{10}(x)$	0.70	1.18	1.48	1.70	1.98
$\log_{10}(y)$	1	1.95	2.56	3	3.56



The data points are fitted by a line that intercepts the $\log(y)$ axis at $\log(A) = -0.40$, so $A = 10^{-0.40} = 0.4$. The gradient can be calculated from $n = (3-1)/(1.70-0.70) = 2$. So the data lie on $y = 0.4x^2$.
Click on the **green** square to return

Solutions to Quizzes

Solution to Quiz: The curve is $y = x^2$. Taking logs of both sides gives: $\log(y) = \log(x^2) = 2\log(x)$, i.e., the log-log plot is a straight line through the origin with gradient 2.

Line b passes though the origin and through the point ($x = 2$, $y = 4$). From the package on **Straight Lines** we know that the gradient, m, of a straight line passing through ($\log(x_1)$, $\log(y_1)$) and ($\log(x_2)$, $\log(y_2)$) is given by

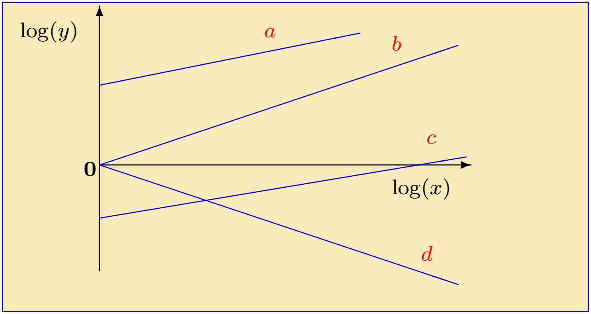
$$m = \frac{\log(y_2) - \log(y_1)}{\log(x_2) - \log(x_1)}$$

we see that the gradient of line b is given by

$$m_b = \frac{4 - 0}{2 - 0} = 2$$

This is therefore the correct log-log plot. End Quiz

Solution to Quiz:



If c corresponds to $y = Ax^n$, then $\log(y) = n\log(x) + \log(A)$. The intercept of line c on the $\log(y)$ axis is negative. This implies that $\log(A) < 0$, which means that $0 < A < 1$. It does **not** signify that A itself is negative. (Of course we also cannot take the logarithm of a negative number like this.)

It may be checked that the other statements are correct.

End Quiz

Solution to Quiz: The equation of a log-log plot is:

$$\log(y) = n\log(x) + \log(A)$$

If we change the base of the logarithm that is used, then the gradient n is unchanged but the intercept, $\log(A)$, is altered.

For example the log-log plot of $y = 3x^4$ in terms of logarithms to the base 10 is:

$$\log_{10}(y) = 4\log_{10}(x) + \log_{10}(3)$$

which has an intercept at $\log_{10}(3) = 0.477$ (to 3 d.p.) Using natural logarithms the equation would become

$$\ln(y) = 4\ln(x) + \ln(3)$$

This has the same gradient, but the intercept on the $\ln(y)$ axis is now at $\ln(3) = 1.099$ (to 3 d.p.)

The only exception to this is if $A = 1$, since $\log_N(1) = 0$ for all N.

End Quiz