Basic Mathematics Revision Pack

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Mathematics Support Series

Basics of Algebra 1

Using Formulae

and

Negative Numbers

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Introduction

Simple formulae exist and are used for a great many different purposes. Any science text book will contain a number of different formulae, and the same is true of many others, such as economics books. Other subject areas technology and also have formulae that are unique to their particular specialism.

A formula can be thought of as simply a way of writing a set of instructions to perform a particular calculation. For example to convert a temperature in Centigrade to Fahrenheit you should multiply by 1.8 and add 32. This could be written as the formula:

$$F = 1.8C + 32.$$

To use formulae effectively you need to be aware of the conventions that are used.

Negative numbers about in real situations, such as if you were overdrawn at the bank, you would have a negative balance. When temperatures drop below zero we use negative numbers and the phrase negative equity affects many home owners in Britain today. To use negative numbers it is important that you understand the basic rules for addition, subtraction, multiplication and division.

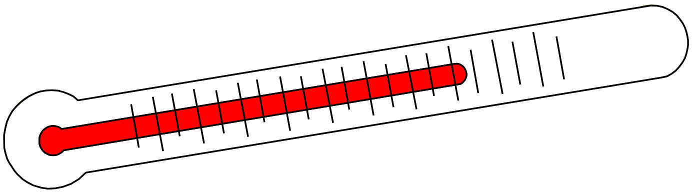
This booklet will be concerned with showing you how to use simple formulae and work with negative numbers. This booklet will help you to:

(i) use simple formulae,

(ii) work with negative numbers,

(iii) use formulae that involve negative numbers.

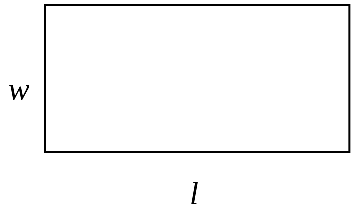
(iv) create simple formulae.



Simple Formulae

In a formula letters are used to represent different numbers or quantities. For example the area of a rectangle can be found using the formula

$$A = wl,$$



where *A* stands for the area, *l* for the length and w for the width. The notation *wl* means that *l* and *w* should be multiplied together. When 2 or more letters are written next to each other they are multiplied together.

The perimeter of the rectangle is represented by the letter *P* and can be calculated using the formula,

$$P = 2w + 2l .$$

Here $2w$ means the width multiplied by 2 and $2l$ the length multiplied by 2. The formula then states that these two numbers should be added to each other.

Example 1

Use the formula $A = wl$ to find the area of a rectangle with a length of 40cm and a width of 20cm. Using the formula with $w = 20$ and $l = 40$, gives

$$\begin{aligned} A &= wl \\ &= 20 \times 40 \\ &= 800\text{cm}^2 . \end{aligned}$$

Example 2

Use the formula $P = 2l + 2w$ to find the perimeter of a rectangle with width 30cm and length 50cm.
Here $w = 30$ and $l = 50$, so using the formula gives

$$\begin{aligned} P &= 2w + 2l \\ &= 2 \times 30 + 2 \times 50 \\ &= 60 + 100 \\ &= 160\text{cm} . \end{aligned}$$

Example 3

The royalties that an author receives from the sales of his book is given by

$$R = \frac{np}{10}$$

(where R = royalties, n = number of copies sold and p = price of the book). This formula states that n and p should be multiplied together and divided by 10.

(a) Find R if 1000 books are sold at a price of £15.

Here n = 1000, p = 15 so,

$$R = \frac{1000 \times 15}{10}$$
$$= £1500 .$$

(b) Find R if 500 books are sold at a price of £12.99.

Here n = 500, p = £12.99 so,

$$R = \frac{500 \times 12.99}{10}$$
$$= £649.50 .$$

Example 4

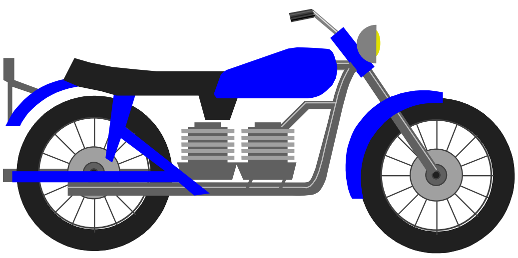
The final speed, v, of a motorbike is given by

$$v = u + at ,$$

where u is the initial speed, a the acceleration and t the time taken.

If u = 10, a = 4, t = 6 find v.

$$v = u + at$$
$$= 10 + 4 \times 6$$
$$= 10 + 24$$
$$= 34\text{ms}^{-1} .$$



Example 5

The cost of printing n leaflets varies between printers. Firm A calculates the cost using the formula

$$C_A = 0.8 + 0.04n$$

and firm B uses the formula

$$C_B = 0.4 + 0.05n\ .$$

(a) Which firm would be best to use if you required 500 leaflets?

Here 500 leaflets are needed so $n = 500$.

Using the formula for firm A

$$\begin{aligned} C_A &= 0.8 + 0.04n \\ &= 0.8 + 0.04 \times 500 \\ &= 0.8 + 20 \\ &= £20.80\ . \end{aligned}$$

Using the formula for firm B gives,

$$\begin{aligned} C_B &= 0.4 + 0.05n \\ &= 0.4 + 0.05 \times 500 \\ &= 0.4 + 25 \\ &= £25.40\ . \end{aligned}$$

So firm A would be cheaper.

(b) Which firm would be best to use if you required 20 leaflets?

In this case $n = 20$.

Using the formula for firm A gives

$$\begin{aligned} C_A &= 0.8 + 0.04 \times 20 \\ &= 0.8 + 0.8 \\ &= £1.60 \end{aligned}$$

Using the formula for firm B gives,

$$\begin{aligned} C_B &= 0.4 + 0.05 \times 20 \\ &= 0.4 + 1.00 \\ &= £1.40\ . \end{aligned}$$

So firm B would be cheaper.

Exercises

1. The formula, $F = 32 + 1.8C$ can be used to convert temperatures in Centigrade to Fahrenheit. F is the temperature in Fahrenheit and C the temperature in centigrade. Find F if;

   (a) $C = 10$,
   (b) $C = 100$,
   (c) $C = 20$.

2. The interest, I, generated by an investment of £P at an interest rate, R, can be calculated using the formula

$$I = \frac{PR}{100} \ .$$

   Find I, giving your answers to the nearest penny, if

   (a) P = £1000 and R = 4.5%,
   (b) P = £50 and R = 3.82%,
   (c) P = £560 and R = 8.2%,
   (d) P = £1027 and R = 4.93%.

3. The length, l, of a metal bar varies as it is heated, and is related to its temperature, T, by the formula

$$l = 9.2 + 0.02T \ .$$

   Find l if

   (a) T = 30°C,
   (b) T = 100°C.

4. The length of a spring is given by

$$l = 15 + 0.02m \ ,$$

   where l is the length in cm and m the mass it supports in grams. Find l if

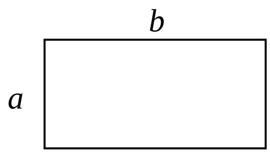
   (a) m = 100 grams,
   (b) m = 2000 grams.

5. (a) A printer charges for printing tickets using the formula

$$C = 5 + 0.04n$$

   where C is the charge and n the number of tickets printed. Find C if n is 20, 40 and 300.

   (b) A second printer charges according to the number of tickets and their size. She uses the formula



$$C = 4 + 0.01nab$$

   where C is the charge, n the number of tickets and a and b the dimensions of the ticket in cm. Find the cost of producing 300 tickets that are 4cm by 6cm.

6. This question concerns the formula,

$$v = u + at\ .$$

Find v if

(a) $u = 20$, $a = 0.5$ and $t = 30$,
(b) $u = 5$, $a = 1.2$ and $t = 10$,
(c) $u = 0$, $a = 1.4$ and $t = 15$.

7. This question concerns the formula,

$$X = 5a + 4bc\ .$$

Find X if;

(a) $a = 6$, $b = 4$ and $c = 2$,
(b) $a = 0$, $b = 2$ and $c = 1$,
(c) $a = 10$, $b = 3$ and $c = 20$.

8. The distance that an object has fallen is given by

$$d = \frac{1}{2}gt^2$$

where t is the time it has been falling and g the acceleration due to gravity. Note that t^2 means t squared or txt. Find d if

(a) the object falls for 10 seconds on the moon, where $g = 1.6\text{ms}^{-2}$,
(b) the object falls for 2 seconds on earth where $g = 9.81\text{ms}^{-2}$.

Solutions

1. (a) $F = 32 + 1.8 \times 10$

$= 50$

(b) $F = 32 + 1.8 \times 100$

$= 212$

(c) $F = 32 + 1.8 \times 20$

$= 68$

2. (a) $I = \frac{1000 \times 4.5}{100}$

$= £45$

(b) $I = \frac{50 \times 3.82}{100}$

$= £1.91$

(c) $I = \dfrac{560 \times 8.2}{100}$

$= £45.92$

(d) $I = \dfrac{1027 \times 4.93}{100}$

$= £50.63$

3. (a) $l = 9.2 + 0.02 \times 30$

$= 9.8$

(b) $l = 9.2 + 0.02 \times 100$

$= 11.2$

4. (a) $l = 15 + 0.02 \times 100$

$= 17$

(b) $l = 15 + 0.02 \times 2000$

$= 55$

5. (a) If $n = 20$; $C = 5 + 0.04 \times 20$
$= £5.80$

If $n = 40$; $C = 5 + 0.04 \times 40$
$= £6.60$

If $n = 300$; $C = 5 + 0.04 \times 300$
$= £17$

(b) $C = 4 + 0.01 \times 300 \times 4 \times 6$

$= £76$

6. (a) $v = 20 + 0.5 \times 30$
$= 35$

(b) $v = 5 + 1.2 \times 10$
$= 17$

(c) $v = 0 + 1.4 \times 15$
$= 21$

7. (a) $X = 5 \times 6 + 4 \times 4 \times 2$
$= 30 + 32$
$= 62$

(b) $X = 5 \times 0 + 4 \times 2 \times 1$
$= 0 + 8$
$= 8$

(c) $X = 5 \times 10 + 4 \times 3 \times 20$
$= 50 + 240$
$= 290$

8. (a) $d = ½ \times 1.6 \times 10^2$
$= 80\text{m}$

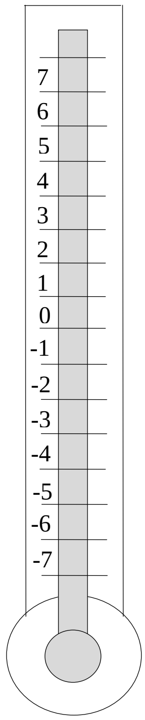
(b) $d = ½ \times 9.81 \times 2^2$
$= 19.62\text{m}$

Working with Negative Numbers

You will be used to working with positive numbers, but often negative numbers have an important role in applications of mathematics. One very obvious example is when dealing with temperatures that drop below zero. Comparing problems with temperatures can often help to explain what is happening.

Addition and Subtraction

Example 1



Negative numbers are produced when a number is taken away from a smaller number;

(a) 6 – 10 = –4
Imagine a temperature of 6°C and then that it cools by 10°C to –4°C .

(b) 4 – 16 = –12
Imagine a temperature of 4°C and then that it cools by 16°C to –12°C .

(c) –4 – 3 = –7
Imagine a temperature of –4°C and then that it cools by 3°C to –7°C.

Exercises

1. 6 – 12 =
2. 5 – 8 =
3. 9 – 20 =
4. 15 – 40 =
5. 6 – 22 =
6. –1 – 6 =
7. –2 – 8 =
8. –10 – 20 =
9. –8 – 4 =
10. 5 – 16 =

Answers

1. –6
2. –3
3. –11
4. –25
5. –16
6. –7
7. –10
8. –30
9. –12
10. –11

Example 2

When adding to negative numbers it is useful to think of temperature increases.

(a) –6 + 10 = 4
Imagine a temperature of –6°C and that it then increases by 10°C to 4°C.

(b) –8 + 3 = –5
Imagine a temperature of –8°C and that it then increases by 3°C to –5°C.

(c) –10 + 4 = –6
Can you explain this in terms of temperature?

Exercises

1.	$-2 + 4 =$	6.	$-8 + 2 =$
2.	$-3 + 10 =$	7.	$-9 + 10 =$
3.	$-6 + 8 =$	8.	$-8 + 6 =$
4.	$-5 + 10 =$	9.	$-100 + 20 =$
5.	$-4 + 8 =$	10.	$-20 + 140 =$

Answers

1.	2	6.	–6
2.	7	7.	1
3.	2	8.	–2
4.	5	9.	–80
5.	4	10.	120

Example 3

Sometimes you may be faced with the problem of adding a negative number, for example $2 + (-4)$. This problem can be overcome by treating it as if you were subtracting a positive number, so,

$$\begin{aligned} 2 + (-4) &= 2 - 4 \\ &= -2 \end{aligned}$$

(a) $2 + (-8) = 2 - 8$ (converting to the subtraction of a positive number)
$= -6$

(b) $3 + (-10) = 3 - 10$ (converting to the subtraction of a positive number)
$= -7$

(c) $-6 + (-4) = -6 - 4$ (converting to the subtraction of a positive number)
$= -10$

Exercises

1.	$2 + (-5) =$	6.	$-6 + (-5) =$
2.	$3 + (-6) =$	7.	$-8 + (-2) =$
3.	$8 + (-4) =$	8.	$-10 + (-4) =$
4.	$9 + (-6) =$	9.	$10 + (-8) =$
5.	$10 + (-12) =$	10.	$6 + (-20) =$

Answers

1.	$2 + (-5) = 2 - 5 = -3$	6.	$-6 + (-5) = -6 - 5 = -11$
2.	$3 + (-6) = 3 - 6 = -3$	7.	$-8 + (-2) = -8 - 2 = -10$
3.	$8 + (-4) = 8 - 4 = 4$	8.	$-10 + (-4) = -10 - 4 = -14$
4.	$9 + (-6) = 9 - 6 = 3$	9.	$10 + (-8) = 10 - 8 = 2$
5.	$10 + (-12) = 10 - 12 = -2$	10.	$6 + (-20) = 6 - 20 = -14$

Example 4

When you subtract a negative number this is equivalent to adding a positive number.

(a) $6 - (-4) = 6 + 4$ (The $-(-4)$ becomes $+ 4$).
$= 10$

(b) $10 - (-5) = 10 + 5$ (The $-(-4)$ becomes $+ 4$).
$= 15$

(c) $-10 - (-10) = -10 + 10$ (The $-(-10)$ becomes $+ 10$).
$= 0$

Imagine someone taking away your overdraft!! The effect of taking away a negative amount such as your overdraft is actually an increase.

Exercises

1. $6 - (-4) =$
2. $7 - (-3) =$
3. $12 - (-5) =$
4. $8 - (-10) =$
5. $14 - (-3) =$
6. $-4 - (-8) =$
7. $-5 - (-3) =$
8, $-6 - (-8) =$
9. $-10 - (-90) =$
10. $-90 - (-10) =$

Answers

1. $6 - (-4) = 6 + 4 = 10$
2. $7 - (-3) = 7 + 3 = 10$
3. $12 - (-5) = 12 + 5 = 17$
4. $8 - (-10) = 8 + 10 = 18$
5. $14 - (-3) = 14 + 3 = 17$
6. $-4 - (-8) = -4 + 8 = 4$
7. $-5 - (-3) = -5 + 3 = -2$
8. $-6 - (-8) = -6 + 8 = 2$
9. $-10 - (-90) = -10 + 90 = 80$
10. $-90 - (-10) = -90 + 10 = -80$

Multiplication and Division

For multiplication and division first ignore the signs and perform the calculation, then give the result a sign according to the rules below:

+	×	+	=	+	+	÷	+	=	+
+	×	–	=	–	+	÷	–	=	–
–	×	+	=	–	–	÷	+	=	–
–	×	–	=	+	–	÷	–	=	+

Example 5

(a) $(-4) \times (-6)$
First multiply 4 by 6 to give 24.
As both numbers are negative the result should be positive, that is +24 or simply 24, so

$$(-4) \times (-6) = 24 .$$

(b) $(-6) \times 3$

First multiply 6×3 to give 18.
As one number is negative and the other positive, the result should be negative, that is –18, so

$$(-6) \times 3 = -18 .$$

(c) $7 \times (-5)$

First multiply 7 by 5 to give 35.
As one number is negative and the other positive, the result will be negative, that is –35, so

$$7 \times (-5) = -35 .$$

(d) $40 \div (-8)$

First divide 40 by 8 to give 5. As one number is positive and the other negative, the result will be negative, that is –5, so

$$40 \div (-8) = -5 .$$

(e) $(-30) \div (-5)$

First divide 30 by 5 to give 6. As both numbers are negative the result will be positive, that is 6, so,

$$(-30) \div (-5) = 6 .$$

Exercises

1. $6 \times (-6) =$
2. $(-7) \times (-2) =$
3. $(-4) \times (-2) =$
4. $3 \times (-5) =$
5. $(-6) \times 2 =$
6. $(-4) \times (-5) =$
7. $3 \times (-4) =$
8. $(-4) \times 10 =$
9. $60 \div (-30) =$
10. $(-18) \div (-6) =$

Answers

1.	$6 \times (-6) = -36$	(Since one negative, one positive)
2.	$(-3) \times (-2) = 14$	(Since both negative)
3.	$(-4) \times (-2) = 8$	(Since both negative)
4.	$3 \times (-5) = -15$	(Since one positive, one negative)
5.	$(-6) \times 2 = -12$	(Since one positive, one negative)
6.	$(-4) \times (-5) = 20$	(Since both negative)
7.	$3 \times (-4) = -12$	(Since one positive, one negative)
8.	$(-4) \times 10 = -40$	(Since one positive, one negative)
9.	$60 \div (-30) = -2$	(Since one positive, one negative)
10.	$(-18) \div (-6) = 3$	(Since both negative)

Exercises

1. Complete the table below, which gives temperature changes.

Initial Temp.	Temp. Change	Final Temp.
5°C	–5°C	
–1°C	+6°C	
–7°C	–4°C	
4°C	–10°C	
7°C	–30°C	
–15°C	+13°C	
–7°C		–10°C
8°C		–2°C
–4°C		–1°C
–13°C		3°C
5°C		–20°C

2. Give the answers to each calculation below:

(a)	–4 + 8 =	(g)	–7 + (–2) =
(b)	–5 – (–6) =	(h)	–4 + 6 =
(c)	(–5) × (–8) =	(i)	(–7) × (–5) =
(d)	(–9) ÷ 2 =	(j)	(–9) × 4 =
(e)	2 × (–5) =	(k)	–11 + (–6) =
(f)	–6 – (–5) =	(l)	(–4) – 9 =

Answers

1. 0, 5, –11, –6, –23, –2, –3, –10, +3, +16, –25.

2.

(a)	4	(g)	–9
(b)	1	(h)	2
(c)	40	(i)	35
(d)	–4½	(j)	–36
(e)	–10	(k)	–17
(f)	–1	(l)	–13

Formulae with Negative Numbers

Example 1

Consider the formula,

$$v = u + at\ .$$

(a) Find v, if $u = 10$, $a = –3$ and $t = 10$.

Substituting these values into the formula gives,

$v = 10 + (–3) \times 10$ (as $(–3) \times 10 = –30$)

$= 10 + (–30)$

$= 10 – 30$ (as $+(–30)$ is simply $–30$)

$= –20$

(b) Find v if $u = –10$, $a = –3$ and $t = 2$.

Substituting these values into the equation gives

$v = –10 + (–3) \times 2$

$= –10 + (–6)$ (as $(–3) \times 2 = –6$)

$= –10 – 6$ (as $+(–6) = –6$)

$= –16$

Example 2

The length, l, of a metal rod varies with temperature, T, according to the formula

$$l = 40 + 0.02T\ .$$

(a) Find the length of the rod when $T = –10°C$.

Substituting this value into the equation gives

$l = 40 + 0.02 \times (–10)$

$= 40 + (–0.2)$ (as $0.02 \times (–10) = –0.2$)

$= 40 – 0.2$ (as $+(–0.2) = –0.2$)

$= 39.8\ .$

(b) Find the length of the rod when $T = –15°C$.

Substituting this value into the equation gives

$l = 40 + 0.02 \times (–15)$

$= 40 + (–0.3)$ (as $0.02 \times (–15) = –0.3$)

$= 40 – 0.3$ (as $+(–0.3) = –0.3$)

$= 39.7\ .$

Example 3

Consider the formula

$$P = 4n - 50 ,$$

which gives the profit made when n items are sold.

(a) Find P if $n = 10$.
Substituting the value for n gives

$$\begin{aligned} P &= 4 \times 10 - 50 \\ &= 40 - 50 \\ &= -10 . \end{aligned}$$

(b) Find P if $n = 12$.
Substituting the value for n gives

$$\begin{aligned} P &= 4 \times 12 - 50 \\ &= 48 - 50 \\ &= -2 . \end{aligned}$$

In both cases the profit is negative, which means that a loss is made.

Exercises

1. The formula below is used for converting temperatures in Centigrade to Fahrenheit, where F is the temperature in Fahrenheit and C the temperature in centigrade.

$$F = 1.80C + 32$$

Convert the temperatures below in centigrade to temperatures in Fahrenheit.

(a) 10°C (c) –10°C
(b) –5°C (d) –20°C

2. Scientists often use temperatures in °Kelvin. The formula below is used for converting temperatures in °Kelvin to °Centigrade. C is the temperature in °Centigrade and K is the temperature in °Kelvin.

$$C = K - 273$$

Convert the temperatures below in Kelvin to Centigrade.

(a) 270°K (c) 275°K
(b) 100°K (d) 150°K

3. Consider the formula,

$$s = \frac{(u+v)t}{2} .$$

Find s if
(a) $u = -4$, $v = 8$, $t = 2$,
(b) $u = 0$, $v = -2$, $t = 10$,

(c) $u = 4$, $v = -10$, $t = 3$,
(d) $u = -1$, $v = -8$, $t = 5$.

4. The length, l, of a spring is given by the formula

$$l = 20 - 0.08F$$

where F is the force applied to the spring to compress it.

Find l if

(a) $F = 5$ (c) $F = 60$
(b) $F = 24$ (d) $F = 180$.

5. The formula

$$P = 100n - 425$$

gives the profit, P, made when n cars are sold in a day at a show room.

Find P if;

(a) $n = 2$,
(b) $n = 4$,
(c) $n = 5$.

How many cars must be sold each day if a loss is not to be made?

6. The average lowest temperature, T, on 3 nights is given by

$$T = \frac{T1 + T2 + T3}{3}$$

where T_1, T_2 and T_3 are the lowest temperatures on each night.

Find T if

(a) $T_1 = -2$, $T_2 = 2$, $T_3 = -3$,

(b) $T_1 = -10$, T_2, $= -4$, $T_3 = 2$,

(c) $T_1 = 0$, $T_2 = -5$, $T_3 = -4$.

7. The temperature gradient, G, in a metal rod is given by

$$G = \frac{T_2 - T_1}{l}$$

where T_1 and T_2 are the temperatures at either end and l is the length of the metal rod.

Find G if

(a) $T_1 = 6$, $T_2 = 2$, $l = 20$,

(b) $T_1 = -2$, $T_2 = -8$, $l = 10$,

(c) $T_1 = 6$, $T_2 = -5$, $l = 15$.

Solutions

1. (a) $F = 1.80 \times 10 + 32$
$= 18 + 32$
$= 50$

(b) $F = 1.80 \times (-5) + 32$
$= -9 + 32$
$= 23$

(c) $F = 1.80 \times (-10) + 32$
$= -18 + 32$
$= 14$

(d) $F = 1.80 \times (-20) + 32$
$= -36 + 32$
$= -4$

2. (a) $C = 270 - 273$
$= -3$

(b) $C = 100 - 273$
$= -173$

(c) $C = 275 - 273$
$= 2$

(d) $C = 150 - 273$
$= -123$

3. (a) $s = ½ \times (-4 + 8) \times 2$
$= ½ \times 4 \times 2$
$= 4$

(b) $s = ½ \times (0 + (-2)) \times 10$
$= ½ \times (-2) \times 10$
$= -10$

(c) $s = ½(4 + (-10)) \times 3$
$= ½(4 - 10) \times 3$
$= ½(-6) \times 3$
$= -9$

(d) $s = ½ \times (-1 + (-8)) \times 5$
$= ½ \times (-1 - 8) \times 5$
$= ½ \times (-9) \times 5$
$= -22.5$

4. (a) $l = 20 - 0.08 \times 5$
$= 20 - 0.4$
$= 19.6$

(b) $l = 20 - 0.08 \times 24$
$= 20 - 1.92$
$= 18.08$

(c) $l = 20 - 0.08 \times 60$
$= 20 - 4.8$
$= 15.2$

(d) $l = 20 - 0.08 \times 180$
$= 20 - 14.4$
$= 5.6$

5. (a) $P = 100 \times 2 - 425$
$= 200 - 425$
$= -225$

(b) $P = 100 \times 4 - 425$
$= 400 - 425$
$= -25$

(c) $P = 100 \times 5 - 425$
$= 500 - 425$
$= 75$

At least 5 cars must be made each day.

6. (a) $T = \dfrac{-2+2+(-3)}{3}$

$= \dfrac{-2+2-3}{3}$

$= -\dfrac{3}{3}$

$= -1$

(b) $T = \dfrac{-10+(-4)+2}{3}$

$= \dfrac{-10-4+2}{3}$

$= -\dfrac{12}{3}$

$= -4$

(c) $T = \dfrac{0+(-5)+(-4)}{3}$

$= \dfrac{-5-4}{3}$

$= -\dfrac{9}{3}$

$= -3$

7. (a) $G = \dfrac{2-6}{20}$

$= -\dfrac{4}{20}$

$= -0.2$

(b) $G = \dfrac{-8-(-2)}{10}$

$= \dfrac{-8+2}{10}$

$= -\dfrac{6}{10}$

$= -0.6$

(c) $G = \dfrac{-5-6}{15}$

$= -\dfrac{11}{15}$

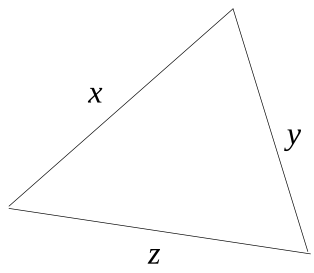
$= -0.733$
(to 3 decimal places).

Creating Formulae

Sometimes you may be required to generate your own formulae. This may be done using the results from an experiment or a set of data, which is considered in Graphs 1. In this section the aim is to create formulae by considering the problem or situation that is presented.

Example 1

Write down a formula to find the perimeter of the triangle shown.



Use the letter P to stand for the perimeter. This is simply obtained by adding together the lengths of the 3 sides, so,

$$P = x + y + z$$

Example 2

At a concert tickets are sold at the prices £5 and £3. Write down a formula for the total value of the ticket sales.
Let T represent the total value, n the number of £5 tickets sold and m the number of £3 tickets sold.
The income from the £3 tickets is 3 times the number of tickets sold, that is $3m$. Similarly the income from the £5 tickets is $5n$. Then the total is obtained by adding these together to give,

$$T = 3m + 5n\ .$$

Example 3

The cost of printing leaflets is made up of a setting up charge of £5 plus a cost of 2p per leaflet. Find a formula for the total cost T of printing n leaflets.

The total is made up of the £5 charge plus the cost of each leaflet multiplied by the number of leaflets. As the total cost is to be in £, then $0.02n$ gives the variable part of the cost. So the total is given by

$$T = 5 + 0.02n\ .$$

Example 4

Write down a formula that can be used to find the mean of the four numbers, w, x, y, z.

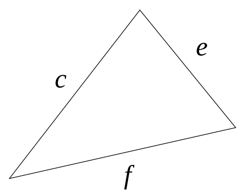
To find the mean, m, the numbers must be added up and divided by 4. So,

$$m = \frac{w + x + y + z}{4}\ .$$

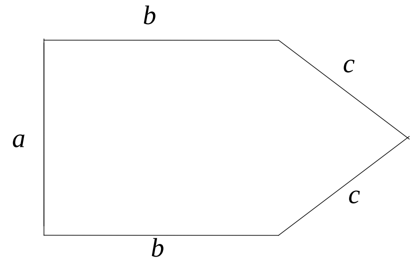
Exercises

1. Write down a formula for the perimeter of each shape below:

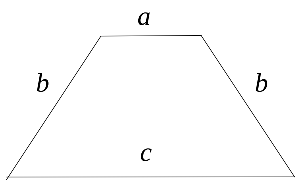
(a)



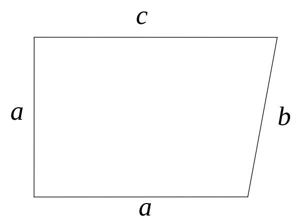
(b)



(c)



(d)



2. Write down a formula to find the mean (average) of the three numbers *p*, *q* and *r*.

3. (a) The cost of producing a component for an engineering company involves a setting up cost of £200 and a production cost of £2 per component. Find a formula for the cost of producing *n* components.

   (b) Repeat (a) for a setting up cost of £150 and £1.50 production cost per component.

4. A computer company sells 2 different models. Model A sells for £950 and model B for £1050. The company sells *m* model A computer and *n* model B computers.

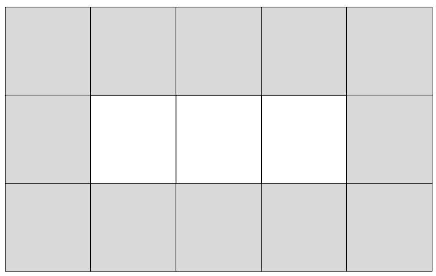
   (a) Find a formula for the total income from the sales of these 2 models.

   (b) If it costs £600 to produce a model A computer and £800 to produce a model B computer, write down a formula for the total cost of producing *m* model A and *n* model B computers.

   (c) Write down a formula for the profit made by selling *m* model A and *n* model B computers.

5. A ticket printer charges a setting up fee of £4 and then 1p for each ticket. Find a formula for the cost of printing *n* tickets.

6. A firm of gardeners create paths in a pattern as shown made up of paving slabs.



   (a) If *n* white slabs are used how many red ones are needed?

   (b) If *n* red slabs are used how many white ones are needed?

Solutions

1. (a) $P = c + e + f$

   (b) $P = a + b + b + c + c$
   $= a + 2b + 2c$

   (c) $P = a + b + b + c$
   $= a + 2b + c$

   (d) $P = a + a + b + c$
   $= 2a + b + c$

2. $m = \dfrac{p+q+r}{3}$

3. (a) $C = 200 + 2n$

   (b) $C = 150 + 1.5n$

4. (a) $I = 950m + 1050n$

   (b) $C = 600m + 800n$

   (c) $P = I - C$

   $= 350m + 250n$

5. $C = 4 + 0.01n$

6. (a) $r = 2n + 6$

   (b) $w = \dfrac{n-6}{2}$

Mathematics Support Series

Basics of Algebra 2

Solving Equations

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Introduction

This unit is concerned with simple equations. Equations very often arise when you attempt to describe a solution mathematically, or sometimes when you use a formula. For example the formula

$$F = 1.8C + 32$$

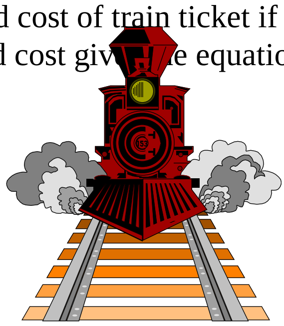
could be used to find F given C, but if you know that $F = 100$ then you have

$$100 = 1.8C + 32.$$

A statement such as this is known as an equation and can arise in many different situations. There are standard techniques for solving such equations.

Equations may also arise when there are certain conditions that specify the value of a quantity indirectly. A very simple example is to find the old cost of train ticket if it has just increased by £4 to £67. Letting the letter x represent the old cost gives the equation

$$x + 4 = 67$$



In this case it is a simple matter to see that x, the old cost is £63. This unit will help you to;

(i) solve simple equations,

(ii) solve equations that arise when using formulae,

(iii) form simple equations.

You will probably find it very helpful to work through the booklet "Basics of Algebra 1, Using Formulae of Negative Numbers" before starting this booklet.

Solving Simple Equations

In this section we will consider equations, which are mathematical expressions containing one unknown term.

Often a formula will lead you to an equation. For example suppose a salesman is paid according to the formula

$$S = 100 + 5n$$

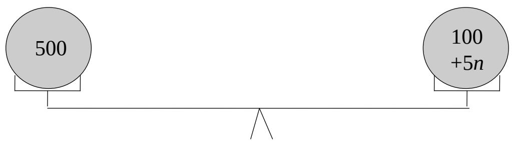
where S is his monthly pay and n the number of items he has sold. How many items must he sell to earn £500?

The formula becomes

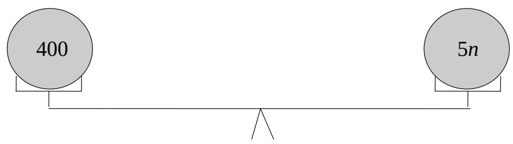
$$500 = 100 + 5n.$$

This is an example of a simple equation, and the task is now to find the value of n.

When dealing with an equation it is useful to think of the 2 sides of the equals sign as being balanced on a set of scales.



In solving the equation the balance must be maintained, so what is done to one side of the equation must also be done to the other. The first step in solving this equation is to remove 100 from either side to give the situation below.



This can be written mathematically as;

$$400 = 5n.$$

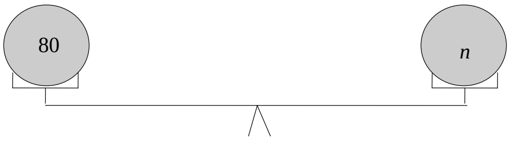
The equation has been simplified by subtracting 100 from each side.

To simplify the equation further the 5 must be removed. To do this both sides must be divided by 5 to give;

$$\frac{400}{5} = n$$

or $\quad n = 80.$

This situation is illustrated below.



The examples below show how to solve a variety of equations using the principles described above. In each case the aim is to find the value of the unknown quantity, represented by a letter. The examples are designed to help you develop skills and techniques, these are then applied to problems in section 2.

Example 1

(a) Solve the equation,

$$x + 6 = 14.$$

To find the value of x, subtract 6 from each side of the equation to give;

$$x + 6 - 6 = 14 - 6.$$

Note that 6 has been added to and subtracted from the left hand side simply leaving x and the right hand side becomes 8, so

$$x = 8.$$

(b) Solve the equation

$$p + 2 = 25.$$

To find the value of p subtract 2 from each side of the equation to give,

$$p + 2 - 2 = 25 - 2$$

or

$$p = 23.$$

(c) Solve the equation,

$$x - 8 = 15.$$

To find the value of x add 8 to both sides of the equation. This cancels the effect of the "-8", to give,

$$x - 8 + 8 = 15 + 8$$

or

$$x = 23.$$

(d) Solve the equation,

$$m - 10 = 3.$$

To find the value of m add 10 to both sides of the equation. This cancels the effect of the "-10", to give,

$$m - 10 + 10 = 3 + 10$$

or

$$m = 13.$$

Exercises

Solve each equation below.

1. $x + 2 = 10$
2. $x + 5 = 16$
3. $x + 4 = 13$
4. $x - 5 = 16$
5. $x - 4 = 13$
6. $x - 8 = 1$
7. $x - 0.2 = 5.3$
8. $q + 0.7 = 1.82$
9. $z - 1.84 = 6.51$
10. $x + 6 = 2$.

Solutions

1. $x = 10 - 2$
   $= 8$

2. $x = 16 - 5$
   $= 11$

3. $x = 13 - 4$
   $= 9$

4. $x = 16 + 5$
   $= 21$

5. $x = 13 + 4$
   $= 17$

6. $x = 1 + 8$
   $= 9$

7. $x = 5.3 + 0.2$
   $= 5.5$

8. $q = 1.82 - 0.7$
   $= 1.12$

9. $z = 6.51 + 1.84$
   $= 8.35$

10. $x = 2 - 6$
    $= -4$.

Example 2

(a) Solve the equation,

$5x = 100.$

When equations involve letters that have been multiplied by a number, as in this example, it is necessary to divide both sides of the equation by that number. So dividing both sides by 5 gives,

$$\frac{5x}{5} = \frac{100}{5} \ .$$

Note that the left hand side has been both multiplied and divided by 5, so the equation becomes,

$x = 20.$

(b) Solve the equation

$6x = 24.$

To remove the 6 both sides of the equation must be divided by 6. This gives,

$$\frac{6x}{6} = \frac{24}{6}$$

or

$x = 4.$

(c) Solve the equation

$$\frac{x}{2} = 9 \cdot$$

Here the letter x has been divided by 2. To remove the 2, both sides of the equation must be multiplied by 2. This gives;

$$\frac{x}{2} \times 2 = 9 \times 2 \ .$$

The left hand side is now just x and the right hand side is just 18, as below,

$x = 18.$

(d) Solve the equation

$$\frac{x}{4} = 5.$$

Here the letter x has been divided by 4, and so both sides of the equation must be multiplied by 4. This gives,

$$\frac{x}{4} \times 4 = 5 \times 4$$

or

$$x = 20.$$

Exercises

Solve each equation below.

1. $4x = 28$
2. $3x = 21$
3. $5x = 110$
4. $\frac{x}{2} = 15$
5. $\frac{x}{8} = 3$
6. $\frac{x}{5} = 12$
7. $5x = 4$
8. $\frac{x}{2.1} = 8.2$
9. $3.2x = 10.4$
10. $8x = 5$

Solutions

1. $x = \frac{28}{4}$
   $= 7$
2. $x = \frac{21}{3}$
   $= 7$
3. $x = \frac{110}{5}$
   $= 22$
4. $x = 15 \times 2$
   $= 30$
5. $x = 3 \times 8$
   $= 24$
6. $x = 12 \times 5$
   $= 60$
7. $x = \frac{4}{5}$
   $= 0.8$
8. $x = 8.2 \times 2.1$
   $= 17.22$
9. $x = \frac{10.4}{3.2}$
   $= 3.25$
10. $x = \frac{5}{8}$
    $= 0.625$

Example 3

(a) Solve the equation,

$$3x + 2 = 14$$

First subtract 2 from both sides of the equation to give,

$$3x = 14 - 2$$
$$3x = 12$$

Now divide both sides of the equation by 3, to give,

$$x = \frac{12}{3}$$

$$= 4$$

(b) Solve the equation

$$5x - 8 = 37.$$

First add 8 to both sides of the equation to give,

$$5x = 37 + 8$$
$$5x = 45.$$

Now divide both sides of the equation by 5 to give;

$$x = \frac{45}{5}$$

$$= 9.$$

(c) Solve the equation,

$$\frac{x}{4} + 18 = 26.$$

First subtract 18 from both sides of the equation to give,

$$\frac{x}{4} = 26 - 18$$

$$\frac{x}{4} = 8.$$

Now multiply both sides of the equation by 4 to give,
$x = 8 \times 4 = 32$.

(d) Solve the equation

$$\frac{x}{3} + 11 = 6$$

First subtract 11 from both sides of the equation to give

$$\frac{x}{3} = 6 - 11$$

$$\frac{x}{3} = -5$$

Now multiply both sides of the equation by 3 to give,

$$x = -5 \times 3$$

$$= -15$$

Exercises

Solve each equation below.

1. $3x + 8 = 23$
2. $5x - 9 = 46$
3. $8x + 12 = 28$
4. $5x - 8 = 13$
5. $6x + 40 = 2$
6. $\frac{x}{4} - 5 = 10$
7. $\frac{x}{2} + 14 = 8$
8. $\frac{x}{3} - 4 = 11$
9. $\frac{x}{6} + 6 = 2$
10. $4x - 8 = -12$

Solutions

1.
$3x = 23 - 8$
$3x = 15$
$x = \frac{15}{3}$
$x = 5$

2.
$5x = 46 + 9$
$5x = 55$
$x = \frac{55}{5}$
$x = 11$

3.
$8x + 12 = 28$
$8x = 16$
$x = \frac{16}{8}$
$x = 2$

4.
$5x = 13 + 8$
$5x = 21$
$x = \frac{21}{5}$
$x = 4.2$

5.
$6x = 2 - 40$
$6x = -38$
$x = \frac{-38}{6}$
$x = -6\frac{1}{3}$

6.
$\frac{x}{4} = 10 + 5$
$\frac{x}{4} = 15$
$x = 15 \times 4$
$x = 60$

7.
$\frac{x}{2} = 8 - 14$
$\frac{x}{2} = -6$
$x = -6 \times 2$
$x = -12$

8.
$\frac{x}{3} = 11 + 4$
$\frac{x}{3} = 15$
$x = 15 \times 3$
$x = 45$

9.
$\frac{x}{6} = 2 - 6$
$\frac{x}{6} = -4$
$x = -4 \times 6$
$x = -24$

10.
$4x = -12 + 8$
$4x = -4$
$x = \frac{-4}{4}$
$x = -1$

Example 4

(a) Solve the equation

$$5(4x + 2) = 110$$

As the 5 multiplies everything in the bracket the first step is to divide both sides of the equation by 5. This gives,

$$\frac{5(4x+2)}{5} = \frac{110}{5}$$

or $4x + 2 = 22$

Next subtracting 2 from both sides gives,

$$4x = 22 - 2$$

$$4x = 20$$

Finally dividing by 4 gives,

$$x = \frac{20}{4}$$

$$x = 5.$$

(b) Solve the equation

$$3(2x - 5) = 15.$$

First both sides should be divided by 3 to give,

$$\frac{3(2x-5)}{3} = \frac{15}{3}$$

or $2x - 5 = 5$

Then adding 5 to both sides of the equation gives

$$2x = 5 + 5$$

$$2x = 10.$$

Finally dividing both sides by 2 give,

$$x = \frac{10}{2}$$

$$x = 5.$$

(c) Solve the equation

$$\frac{4x-5}{3} = 9$$

As the whole of the left had side is divided by 3, both sides of the equation must first be multiplied by 3. This gives,

$$3 \times \left(\frac{4x-5}{3}\right) = 3 \times 9$$

or $4x - 5 = 27.$

Now 5 can be added to both sides to give

$$4x = 27 + 5$$

$$4x = 32.$$

Finally dividing both sides by 4 gives,

$$x = \frac{32}{4}$$

$$x = 8.$$

(d) Solve the equation

$$\frac{4x}{5} + 6 = 14 \cdot$$

The first stage here is to remove the 6. This is done by subtracting 6 from both sides to give

$$\frac{4x}{5} = 14 - 6$$

$$\frac{4x}{5} = 8.$$

Both sides can now be multiplied by 5 to give,

$4x = 8 \times 5$

$4x = 40.$

Finally both sides can be divided by 4 to give,

$x = \dfrac{40}{4}$

$x = 10.$

Exercises

Solve each equation below:

1. $3(x + 2) = 21$
2. $5(2x - 1) = 45$
3. $6(4x\text{-}5) = 18$
4. $2(x + 4) = 9$
5. $\dfrac{(x-8)}{5} = 3$
6. $\dfrac{2x+70}{4} = 11$
7. $\dfrac{4x}{3} + 6 = 14$
8. $\dfrac{3x}{4} - 12 = 3$
9. $\dfrac{5(x+4)}{2} = 15$
10. $\dfrac{6(5x-8)}{4} = 12$

Solutions

1. $x + 2 = \dfrac{21}{3}$

$x + 2 = 7$

$x = 7 - 2$

$x = 5$

2. $2x - 1 = \dfrac{45}{5}$

$2x - 1 = 9$

$2x = 9 + 1$

$2x = 10$

$x = \dfrac{10}{2}$

$x = 5$

3. $4x - 5 = \dfrac{18}{6}$

$4x - 5 = 3$

$4x = 3 + 5$

$4x = 8$

$x = \frac{8}{4}$

$x = 2$

4. $x + 4 = \frac{9}{2}$

$x = 4.5 - 4$

$x = 4.5 - 4$

$x = 0.5$

5. $x - 8 = 3 \times 5$

$x - 8 = 15$

$x = 15 + 8$

$x = 23$

6. $2x + 70 = 11 \times 4$

$2x + 70 = 44$

$2x = 44 - 70$

$2x = -26$

$x = \frac{-26}{2}$

$x = -13.$

7. $\frac{4x}{3} = 14 - 6$

$\frac{4x}{3} = 8$

$4x = 8 \times 3$

$x = \frac{24}{4}$

$x = 6$

8. $\frac{3x}{4} = 3 + 12$

$\frac{3x}{4} = 15$

$3x = 15 \times 4$

$3x = 60$

$x = \frac{60}{3}$

$x = 20$

9. $5(x + 4) = 15 \times 2$

$5(x + 4) = 30$

$x + 4 = \frac{30}{5}$

$x + 4 = 6$

$x = 6 - 4$

$x = 2$

10. $6(5x - 8) = 12 \times 4$

$6(5x - 8) = 48$

$5x - 8 = \frac{48}{6}$

$5x - 8 = 8$

$5x = 8 + 8$

$5x = 16$

$x = \frac{16}{5}$

$x = 3.2.$

Equations From Formulae

When substituting values into a formula the result is often in the form of an equation. Examples of this type are considered in this section.

Example 1

Consider the formula $v = u + at$. Find a if $v = 10$, $u = 2$ and $t = 4$

Substituting these values into the formula gives,

$$10 = 2 + a \times 4.$$

or $$10 = 2 + 4a.$$

To solve this equation first subtract 2 from each side to give,

$$10 - 2 = 4a$$

$$8 = 4a.$$

Then divide both sides by 4 to give

$$\frac{8}{4} = a$$

$$2 = a$$

or $$a = 2.$$

Example 2

The circumference of a circle is given by the formula $C = 2\pi r$. Find the radius of a circle with a circumference of 12cm. Note that you will probably have a π button on your calculator. If not use $\pi = 3.14$.

Substituting the values into the formula gives,

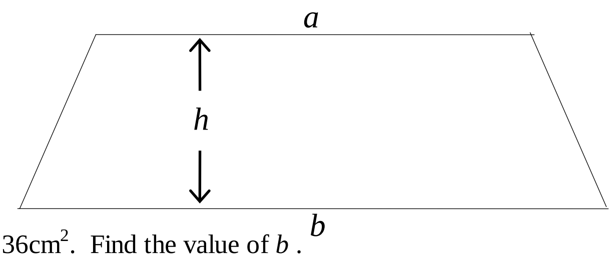
$$12 = 2\pi r.$$

Dividing both sides of the equation by 2π gives,

$$\frac{12}{2\pi} = r$$

or $$r = 1.91 \text{ (to 2 decimal places).}$$

Example 3



The area of a trapezium is given by

$$A = \frac{1}{2}h(a+b).$$

A trapezium has h = 6cm, a = 4cm and an area of 36cm^2. Find the value of b .

Substituting the values given into the formula gives;

$$36 = \frac{1}{2}\times 6(4+b).$$

This can be simplified to,

$$36 = 3\,(4 + b).$$

Now both sides can be divided by 3 to give,

$$\frac{36}{3} = 4+b$$

$$12 = 4 + b.$$

Finally subtracting 4 gives,

$$12 - 4 = b$$

$$8 = b$$

or $b = 8$.

Exercises

1. Ohm's law is often stated in the form,

$$V = IR.$$

   (a) Find V if I = 6 and R = 100.
   (b) Find R if I = 0.01 and V = 2.
   (c) Find I if V = 100 and R = 80.

2. Consider the formula $v = u + at$.

   (a) Find u if v = 10, a = 0.2 and t = 20.
   (b) Find a if v = 6, u = 12 and t = 10.
   (c) Find t if v = 40, u = 10 and a = 0.3.

3. The formula below is used for converting temperatures in centigrade to fahrenheit.

$$F = 32 + 1.8C$$

(a) Convert 80°C into fahrenheit.
(b) Convert 200°F into centigrade.
(c) Convert 150°F into centigrade.

4. The length of a spring supporting a mass is given by:

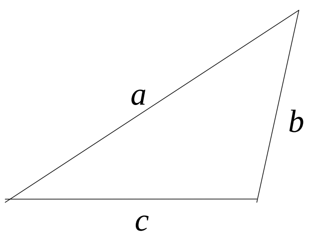
$$L = 20 + 0.2m.$$

Find m if:

(a) $L = 24$ (b) $L = 26.4$

5. The perimeter of a triangle is,

$$P = a + b + c.$$



(a) Find c if $P = 12$, $a = 3$ and $b = 4$.
(b) Find b if $P = 16$, $a = 8$ and $c = 6$.

6. The weekly salary of a salesman is given by

$$S = 40 + 5n$$

where n is the number of sales made.

Find the number of sales needed for a weekly salary if:

(a) £150 (b) £100.

7. Consider the equation

$$s = \frac{1}{2}(u + v)t$$

(a) Find t if $s = 100$, $u = 20$ and $v = 30$.
(b) Find v if s = 200, $u = 15$ and $t = 35$.

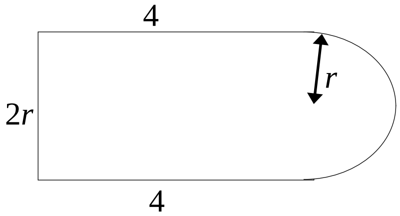
8. The average of three temperatures, T_1, T_2 and T_3, is given as;

$$T = \frac{T_1 + T_2 + T_3}{3}$$

(a) If $T = 21$, $T_1 = 19$ and $T_2 = 24$, find T_3.
(b) If $T = -2$, $T_1 = -8$ and $T_2 = -6$, find T_3.

9. The average of four numbers is 12. If three of the numbers are, 9, 15, and 10, find the other number.

10. The perimeter of the shape shown is ,



$P = 8 + 2r + 2\pi r$.

(a) Find P if $r = 2$.
(b) Find r if $P = 20$.

11. Using the formula $I = \frac{PR}{100}$ used for calculating interest, find

(a) P if I = £56 and R = 8%,
(b) R if I = £1120 and P = £20,000,
(c) P if I = £7000 and R = 7.4%.

12. Using the formula $1 = 9.2 + 0.02T$ from question 3, find T if

(a) $1 = 9.28$ (b) $1 = 9.31$ (c) $1 = 9.18$.

13. The profits made by a company on the sales of a particular item are given by $P = 84n - 6500$, where n is the number of items sold.

(a) Find P if $n = 1000$.
(b) How many items must be sold to give a profit of £2000.
(c) How many items must be sold to break even (ie. $P = 0$).

14. In weight lifting contests the formula below is sometimes used to calculate a "handicapped weight",

$H = L - W$

where H is handicapped left, L is actual weight lifted and W is the weight of the contestant.

(a) Find H if a contestant of weight 80kg lifts 180kg.
(b) What weight must a contestant of weight 75kg lift to get H = 65kg.
(c) A contestant gets H = 115kg by lifting 180kg. What is his weight?

Solutions

1. (a) $V = 6 \times 100$
$= 600$

(b) $2 = 0.01R$

$\frac{2}{0.01} = R$

$R = 200.$

(c) $100 = 80I$

$\frac{100}{80} = I$

$I = 1.25$

2. (a) $10 = u + 0.2 \times 20$
$10 = u + 4$
$10 - 4 = u$
$u = 6$

(b) $6 = 12 + 10a$

$6 - 12 = 10a$

$-6 = 10a$

$\frac{-6}{10} = a$

$a = -0.6$

(c) $40 = 10 + 0.3t$

$40 - 10 = 0.3t$

$30 = 0.3t$

$\frac{30}{0.3} = t$

$t = 100.$

3. (a) $F = 32 + 1.8 \times 80$

$= 176$

(b) $200 = 32 + 1.8C$

$200 - 32 = 1.8C$

$168 = 1.8C$

$\frac{168}{1.8} = C$

$C = 93\frac{1}{3}$

(c) $150 = 32 + 1.8C$

$150 - 32 = 1.8C$

$118 = 1.8C$

$\frac{118}{1.8} = C$

$C = 65.56$
(to 2 decimal places).

4. (a) $24 = 20 + 0.2m$

$4 = 0.2m$

$\frac{4}{0.2} = m$

$m = 20.$

(b) $26.4 = 20 + 0.2m$

$26.4 - 20 = 0.2m$

$6.4 = 0.2m$

$\frac{6.4}{0.2} = m$

$m = 32.$

5. (a) $12 = 3 + 4 + c$
$12 = 7 + c$
$12 - 7 = c$
$c = 5$

(b) $16 = 8 + b + 6$
$16 = 14 + b$
$16 - 14 = b$
$b = 2.$

6. (a) $150 = 40 + 5n$

$150 - 40 = 5n$

$110 = 5n$

$\frac{110}{5} = n$

$n = 22$

(b) $100 = 40 + 5n$

$100 - 40 = 5n$

$60 = 5n$

$\frac{60}{5} = n$

$n = 12$

7. (a) $100 = \frac{1}{2}(20 + 30)t$

$100 = 25t$

$\frac{100}{25} = t$

$t = 4.$

(b) $200 = \frac{1}{2}(15 + v) \times 35$

$200 = 17.5\,(15 + v)$

$\frac{200}{17.5} = 15 + v$

$11.43 = 15 + v$

$11.43 - 15 = v.$

$v = -3.57$

(to 2 decimal places)

8. (a) $21 = \frac{19 + 24 + T_3}{3}$

$21 \times 3 = 43 + T_3$

$63 - 43 = T_3$

$T_3 = 20$

(b) $-2 = \frac{-8 + (-6) + T_3}{3}$

$-2 \times 3 = -14 + T_3$

$-6 + 14 = T_3$

$T_3 = 8$

9. Let x be the unknown number.

$12 = \frac{9+15+10+x}{4}$

$12 \times 4 = 34 + x$

$48 - 34 = x$

$x = 14$

10. (a) $P = 8 + 2 \times 2 + 2\pi \times 2$

$= 8 + 4 + 12.56$

$= 24.56$

(b) $20 = 8 + 2r + 2\pi r$

$20 - 8 = 2r + 6.28r$

$12 = 8.28r$

$\frac{12}{8.28} = r$

$r = 1.449.$

11. (a) $56 = \frac{8P}{100}$

$5600 = 8P$

$\frac{5600}{8} = P$

$P = 700$

(b) $1120 = \frac{20000R}{100}$

$112000 = 20000R$

$\frac{112000}{20000} = R$

$R = 5.6$

(c) $7000 = \frac{7.4P}{100}$

$700000 = 7.4P$

$\frac{700000}{7.4} = P$

$P = 94594.59$

12. (a) $9.28 = 9.2 + 0.02T$

$0.08 = 0.02T$

$\frac{0.08}{0.02} = T$

$T = 4$

(b) $9.31 = 9.2 + 0.02T$

$0.11 = 0.02T$

$\frac{0.11}{0.02} = T$

$T = 5.5$

(c) $9.18 = 9.2 + 0.02T$

$-0.02 = 0.02T$

$\frac{-0.02}{0.02} = T$

$T = -1$

13. (a) $P = 84 \times 1000 - 6500$

$= 77500$

(b) $2000 = 84n - 6500$

$8500 = 84n$

$\frac{8500}{84} = n$

$n = 101.2.$

102 needed for a profit of just over £2000.

(c) $0 = 84n - 6500$

$6500 = 84n$

$\frac{6500}{84} = n$

$n = 77.38$

78 needed to break even.

14. (a) $H = 180 - 80$

$= 100$

(b) $65 = L - 75$

$L = 140$kg

(c) $115 = 180 - W$

$115 + W = 180$

$W = 180 - 115$

$= 65$kg.

Further Techniques for Solving Equations

Sometimes you may be faced with equations that are not like the simple examples considered so far. Some for example may have letters on both sides or have a negative number multiplying the letter. Examples of these types are considered in this section. Also considered are examples where a number is divided by a letter.

Example 1

(a) Solve the equation

$$6 - 2x = 5.$$

A good approach when faced with an equation like this, with a negative number multiplying x is to add $2x$ to both sides, to give

$$6 - 2x + 2x = 5 + 2x$$

or $$6 = 5 + 2x.$$

Then subtracting 5 from both sides, gives

$$1 = 2x$$

and dividing by 2 gives,

$$\frac{1}{2} = x$$

or $$x = \frac{1}{2}.$$

(b) Solve the equation

$$10 = 5 - 8x.$$

A good way to start is to add $8x$ to both sides to give,

$$10 + 8x = 5.$$

The solution can then be obtained by subtracting 10 from both sides, to give,

$$8x = -5$$

and then dividing by 8 to give,

$$x = \frac{-5}{8}$$

$$= -0.625.$$

Example 2

(a) Solve the equation,

$$6x + 5 = 3x - 7$$

To tackle an equation like this it is first necessary to remove all the x's from one side. In this example this can be done by subtracting $3x$ from both sides, which gives,

$$6x + 5 - 3x = -7$$

$$3x + 5 = -7$$

Then subtracting 5 from both sides gives,

$$3x = -12.$$

Finally dividing both sides by 3 gives,

$$x = \frac{-12}{3}$$

$$x = -4.$$

(b) Solve the equation

$$16y + 87 = 108 - 5y$$

Begin by adding $5y$ to both sides to give,

$$21y + 87 = 108.$$

Then subtract 87 from both sides to give,

$$21y = 21$$

and then divide by 21 to give,

$$y = \frac{21}{21}$$

$$y = 1 .$$

Example 3

(a) Solve the equation

$$\frac{4}{x} = 50$$

The first step here is to multiply both sides of the equation by x. This gives,

$$\frac{4}{x} \times x = 50 \times x$$

or $4 = 50x$.

It is now a simple matter to find x by dividing both sides by 50, to give,

$$\frac{4}{50} = x$$

$$x = 0.08.$$

(b) Solve the equation

$$\frac{x+5}{x} = 8\,.$$

Again begin by multiplying both sides of the equation by x, to give,

$$x + 5 = 8x.$$

Now as x appears on both sides, subtract x from both sides to give,

$$5 = 7x.$$

Finally divide by 7 to give,

$$x = \frac{5}{7}.$$

(c) Solve the equation,

$$\frac{4-3x}{2x} = 5.$$

First multiply both sides by $2x$ to give,

$$4 - 3x = 10x.$$

Now add $3x$ to both sides to give,

$$4 = 13x$$

and now divide by 13 to give,

$$x = \frac{4}{13}\ .$$

Exercises

Solve each equation below.

1. $4 - 5x = -11$
2. $6 - 2x = -4$
3. $3x + 4 = 8x - 5$
4. $5x - 2 = 6x + 10$
5. $10 - 4x = 2x + 3$
6. $6x - 9 = 5 - 2x$
7. $8x - 4 = 4 - 3x$
8. $4(x - 2) = 16x + 12$
9. $\frac{16}{x} = 20$
10. $\frac{x-5}{x} = 16$
11. $\frac{8x-2}{4x} = 10$
12. $\frac{5-3x}{3x} = 4$

Solutions

1. $4 = -11 + 5x$

   $15 = 5x$

   $\frac{15}{5} = x$

   $x = 3$

2. $6 = -4 + 2x$

   $10 = 2x$

   $\frac{10}{2} = x$

   $x = 5$

3. $4 = 5x - 5$

   $9 = 5x$

   $\frac{9}{5} = x$

   $x = 1.8$

4. $-2 = x + 10$

$-12 = x$

$x = -12$

5. $10 = 6x + 3$

$7 = 6x$

$\frac{7}{6} = x$

6. $8x - 9 = 5$

$8x = 14$

$x = \frac{14}{8}$

$x = 1.75$

7. $11x - 4 = 4$

$11x = 8$

$x = \frac{8}{11}$

8. $x - 2 = \frac{16x + 12}{4}$

$x - 2 = 4x + 3$

$-2 = 3x + 3$

$-5 = 3x$

$\frac{-5}{3} = x$

$x = -\frac{5}{3}.$

9. $16 = 20x$

$\frac{16}{20} = x$

$x = \frac{3}{4}$

10. $x - 5 = 16x$

$-5 = 15x$

$\frac{-5}{15} = x$

$x = -\frac{1}{3}$

11. $8x - 2 = 40x$

$-2 = 32x$

$-\frac{2}{32} = x$

$x = -\frac{1}{16}$

12. $5 - 3x = 12x$

$5 = 15x$

$\frac{5}{15} = x$

$x = \frac{1}{3}$

Forming Equations

Situations may arise where you need to write a simple equation to solve a problem to find an unknown quantity. The approach is to use a letter to represent the unknown quantity and then to write down an equation that it must satisfy. The examples below will illustrate this purpose.

Example 1

A customer asks to buy 8 tickets for a production at the theatre box office. Find the cost of 1 ticket, if the total cost is £53 and includes an advance booking charge of £3.

Solution

Let n represent the number of tickets, then $8n$ will the total cost of the tickets. The total paid will then be $8n + 3$ which includes the booking charge of £3. Now we can write down the equation

$$8n + 3 = 53$$

as the total cost is £53. Solving the equation gives;

$$8n = 50$$

$$n = \frac{50}{8}$$

$$= £6.25.$$

Example 2

The cost of a hire car is based on a fixed charge of £22 per day plus a charge of 5p per mile. A company receive a bill for £100 for a car used on two consecutive days by an employee. Find the number of miles travelled by the employee.

Solution

Let m represent the number of miles travelled. Then the total cost is made of the mileage charge of $0.05m$ plus the fixed charge for 2 days of £44 to give a total of £100. So the equation is:

$$44 + 0.05m = 100$$

This can now be solved

$$0.05m = 56$$

$$m = \frac{56}{0.05}$$

$$= 1120 \text{ miles.}$$

Example 3

A taxi firm charge £2 for the first mile and £1.20 for each subsequent mile. If a fare is £11.60, find the distance travelled.

Solution

It is easier to think of the pricing policy as £1.20 per mile plus an extra 80p for the first mile. If n is the number of miles then the cost is given by $1.20n + 0.80$. So for a fare of £11.60 we have the equation.

$$11.60 = 1.20n + 0.80$$

Solving this gives,

$$10.80 = 1.20n$$

$$n = \frac{10.80}{1.20}$$

$$= 9 \text{ miles.}$$

Example 4

When three consecutive numbers are added together, their total is 120. Find the three numbers.

Solution

If the lowest number is x, then the next will be $x + 1$ and the highest will be $x + 2$. These three together must give 120, so

$$x + x + 1 + x + 2 = 120$$

or

$$3x + 3 = 120$$

This equation can now be solved to give,

$$3x = 117$$

$$x = \frac{117}{3}$$

$$= 39.$$

So the three numbers are 39, 40 and 41.

Exercises

1. The cost of hiring a photocopier is based on a daily charge of £25 and a charge of 8p per copy. A firm receives a bill for £171.96, for use over a 5 day period. Form an equation and find the number of copies made.

2. A survey report on traffic flow claims that the number of cars travelling per hour on a particular road between 4 and 5pm is 208 more than between 3 and 4pm. Also the number between 5 and 6pm is double the number between 4 and 5pm. Form an equation and find the number of cars on the road between 3 and 4pm, if a total of 1752 cars pass along the road.

3. Five consecutive numbers are added together to give a number that is seven times the first number. Form a suitable equation and find the five numbers.

4. If 3 consecutive even numbers have a total of 204 when added together find the 3 numbers.

5. A train travels at a constant speed of 60mph and then travels at 50mph for the same time. If the journey travels a total of 330 miles find the time taken for the journey.

6. A man walks at 4ms^{-1} for a period of time and then runs 7ms^{-1} for twice as long. How long does it take him to travel 720m.

7. The cost of buying glass is £12 per square metre, plus a cutting charge of £1.50. The cost of a sheet of glass with a width of 1.2m is £10.14. Find the length of the sheet of glass.

8. The length of a snooker table is twice its width. If the area of a table is 4m^2, find the dimensions of the table.

9. Paper such as A3, A4 etc. are cut so that the length is 1.41 times the width. If the area of a sheet of this type of paper is approximately 4990cm^2. Find the dimension of the sheet.

10. The cost of hiring a cement mixer is calculated on a daily basis. The charge is reduced by £2 each day. Find the cost for the first day if it has been hired for 5 days, the total cost is £105.

Solutions

1. Let n = number of copies

   then $171.96 = 125 + 0.08n$

   $46.96 = 0.08n$

   $n = 587$

2. Let n = number in first hour,

   then $n + 208$ = number in second hour

   and $2n + 416$ = number in last hour

   $n + n + 208 + 2n + 416 = 1762$

   $4n + 624 = 1752$

   $4n = 1128$

   $n = 282.$

3. Let x = first number,

   $x + 1$ = second number,

   $x + 2$ = third number, etc.

   $x + x + 1 + x + 2 + x + 3 + x + 4 = 7x$

   $5x + 10 = 7x$

   $10 = 2x$

   $x = 5.$

   The numbers are 5, 6, 7, 8, 9

4. Let x = first number,

   $x + 2$ = second number,

   $x + 4$ = third number.

   $x + x + 2 + x + 4 = 204$

   $3x + 6 = 204$

   $3x = 198$

   $x = 66.$

The numbers are 66, 68 and 70.

5. Let t = time

then $50t$ = distance in first half

and $60t$ = distance in second half

so $50t + 60t = 330$

$110t = 330$

$t = 3$

Time for journey is 6 hours.

6. Let t = time for walking

then $4t$ = distance walked

also $2t$ = time for running

then $14t$ = distance run

$4t + 14t = 720$

$18t = 720$

$t = 40$

Time taken is 120 seconds or 2 minutes.

7. Let x = length of sheet

Then $12 \times 1.2 \times x + 1.50 = 10.14$

$14.4x + 1.50 = 10.14$

$14.4x = 8.64$

$x = 0.6\text{m}$

8. Let w = width

then $2w$ = length

$2w \times w = 4$

$w^2 = 2$

$w = 1.41$

Dimensions 1.41×2.82m.

9. Let w = width

then $1.41w$ = length

$1.41w \times w = 4990$

$$w^2 = \frac{4990}{1.41}$$

$w = 59.5$cm

$l = 83.9$cm.

10. Let x = cost of first day

then $x - 2$ = cost of second day, etc.

$$x + x - 2 + x - 4 + x - 6 + x - 8 = 105$$

$$5x - 20 = 105$$

$$5x = 125$$

$$x = 25$$

Mathematics Support Series

Basics of Algebra 3

Rearranging Formulae

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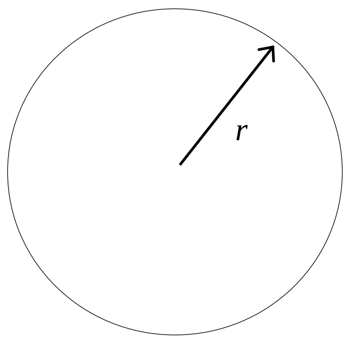
Introduction

This unit is concerned with the rearrangement of formulae. You may have encountered a situation where by using a formula you repeatedly find that you generate equations to solve. It is possible to rearrange most equations to a form that is more conventient for you to use.

The formula $C = 2\pi r$ is used to find the circumference of a circle given the radius. Here C is the "subject" of the formula. It can be rearranged in the form $r = \frac{C}{2\pi}$ to find the radius given the circumference. In this form r is the subject of the formula. This booklet aims to demonstrate how to change the subject of a formula.

You may find it useful to have studied the booklet "Solving Equations" before starting this booklet.



Rearranging Formulae

Often a formula like

$$v = u + at$$

may be useful if it can be rewritten in the form

$$a = \ldots\ldots$$

Changing formulae in this way is known as either transposition or rearrangement of formulae. The approach used is very similar to that used for solving equations. In the example above v is the subject of the original formula, which is rearranged so that a becomes the subject.

Example 1

Rearrange the formula $V = IR$ so that it is in the form $I = \ldots\ldots$

To deal with this problem, divide both sides of the formula by R, which gives

$$\frac{V}{R} = \frac{IR}{R}$$

or $$\frac{V}{R} = I$$

$$I = \frac{V}{R}.$$

Example 2

Rearrange the formula, $v = u + at$, so that it is in the form $t = \ldots\ldots$

First subtract u from both sides of the equation, which gives,

$$v - u = u + at - u$$

or $$v - u = at.$$

Now both sides of the formula can be divided by a, which gives

$$\frac{v-u}{a} = \frac{at}{a}$$

or $$\frac{v-u}{a} = t$$

so that

$$t = \frac{v-u}{a}.$$

Example 3

Make n the subject of the formula

$$P = 50 + 6n\ .$$

The first step is to subtract 50 from both sides to give

$$P - 50 = 6n\ .$$

Then both sides can be divided by 6, which gives

$$\frac{P-50}{6} = n$$

or $$n = \frac{P-50}{6}\ .$$

Example 4

The formula

$$F = 32 + 1.8C$$

is used for converting temperatures from Centigrade to Fahrenheit. Make C the subject to create a formula for converting Fahrenheit to Centigrade. Begin by subtracting 32 from both sides, which gives

$$F - 32 = 1.8C\ .$$

Then divide by 1.8 to give

$$\frac{F-32}{1.8} = C$$

or $$C = \frac{F-32}{1.8}\ .$$

Example 5

The results of an experiment give the formula

$$T = 30(1 + 0.5x)\ .$$

Make x the subject of the formula.

First dividing both sides by 30 gives

$$\frac{T}{30} = 1 + 0.5x\ .$$

Then subtracting 1 from each side gives

$$\frac{T}{30} - 1 = 0.5x\ .$$

Now both sides can be divided by 0.5 <u>or</u> multiplied by 2. Explain why it is better to multiply by 2. This gives

$$2\left(\frac{T}{30}-1\right)=x$$

or $$x=2\left(\frac{T}{30}-1\right).$$

Exercises

1. Make n the subject of the formula, $S = 100 + 5n$.

2. Make u the subject of the formula, $v = u + at$.

3. The perimeter of a rectangle is given by

$$P = 2(l + w)\ .$$

Make w the subject of the formula.

4. For the formula $C = (50 + 30n)m$

(a) make n the subject,
(b) make m the subject.

5. Rearrange each formula below in the form specified.

(a) $v = u + at$
$t = ...$

(b) $1 = 8.7 + 0.15T$
$T = ...$

(c) $P = 5000 - 6n$
$n = ...$

(d) $I = \dfrac{PR}{100}$
R = ...

(e) $C = 0.8 + 0.04n$
$n = ...$

(f) $p = 2(1 + w)$
$1 = ...$

(g) $V = 2\pi rh$
$h = ...$

(h) $F = 32 + 1.8(k - 273)$

6. A persons salary S is related to the tax, T, then by the formula,

$$T = 0.25(S - 5000)$$

provided that S is greater than £5000.

(a) Find T if S = 10,000.
(b) Find S if T = 2,500.
(c) Find a formula for S in terms of T.

Solutions

1. $S - 100 = 5n$

$\frac{S-100}{5} = n$

$n = \frac{S-100}{5}$

2. $v - at = u$

$u = v - at$

3. $\frac{P}{2} = l + w$

$\frac{P}{2} - l = w$

$w = \frac{P}{2} - l$

4. (a) $\frac{C}{m} = 50 + 30n$

$\frac{C}{m} - 50 = 30n$

$\frac{\left(\frac{C}{m} - 50\right)}{30} = n$

$n = \frac{\left(\frac{C}{m} - 50\right)}{30}$

(b) $\frac{C}{(50+30n)} = m$

$m = \frac{C}{(50+30n)}$

5. (a) $v - u = at$

$\frac{v-u}{a} = t$

$t = \frac{v-u}{a}$

(b) $l - 8.7 = 0.15T$

$\frac{l-8.7}{0.15} = T$

$T = \frac{l-8.7}{0.15}$

(c) $P + 6n = 5000$

$6n = 5000 - P$

$n = \frac{5000-P}{6}$

(d) $100I = PR$

$\frac{100I}{P} = R$

$R = \frac{100I}{P}$

(e) $C - 0.8 = 0.04n$

$\frac{C-0.8}{0.04} = n$

$n = \frac{C-0.8}{0.04}$

(f) $\frac{P}{2} = l + w$

$\frac{P}{2} - w = l$

$l = \frac{P}{2} - w$

(g) $V = 2\pi rh$

$$\frac{V}{2\pi r} = h$$

$$h = \frac{V}{2\pi r}$$

(h) $F - 32 = 1.8(k - 273)$

$$\frac{F-32}{1.8} = k - 273$$

$$\frac{F-32}{1.8} + 273 = k$$

$$k = \frac{F-32}{1.8} + 273$$

6. (a) $T = 0.25\ (10{,}000 - 5000)$
$= 1250$

(b) $2500 = 0.25\ (S - 5000)$

$$\frac{2500}{0.25} = S - 5000$$

$$10000 + 5000 = S$$

$$15000 = S$$

$$S = 15000$$

(c) $4T = S - 5000$

$$4T + 5000 = S$$

$$S = 4T + 5000$$

Further Examples

This section will consider further techniques that can be used when rearranging formula. These include examples with squares and square roots as well as more fractions.

Example 1

Formulae of the form

$$\frac{1}{x}=\frac{1}{y}+\frac{1}{z}$$

often arise in science. Make y the subject of this expression.

Solution

First multiply every term of the expression by xyz. This will remove all the fractions, to give,

$$\frac{xyz}{x}=\frac{xyz}{y}+\frac{xyz}{z}$$

or

$$yz=xz+xy$$

Now bring all the terms involving y to the left hand side to give,

$$yz-xy=xz$$

As y appears in both terms on the left hand side it can be written as;

$$y(z-x)=xz$$

and now dividing by $(z-x)$ gives,

$$y=\frac{xz}{z-x}$$

Example 2

Make q the subject of the formula.

$$p=\frac{1}{q}+v$$

Solution

Firstly multiply every term of the expression by q to give,

$$pq = \frac{q}{q} + qv$$

or

$$pq = 1 + qv$$

Now bring all the terms contain q to the left hand side to give,

$$pq - pv = 1$$

Because q appears in both terms on the left hand side it can be written as,

$$q(p - v) = 1$$

Now dividing by $(p - v)$ gives,

$$q = \frac{1}{p - v}$$

Example 3

Make t the subject of the formula.

$$F = \frac{mv - mu}{t}$$

Solution

Firstly multiply both sides by t to give,

$$Ft = mv - mu.$$

Now divide by F to give,

$$t = \frac{mv - mu}{F}$$

Example 4

Make x the subject of the formula,

$$q = \sqrt{\frac{x - p}{2}}$$

Solution

The first step is to remove the square root, by squaring both sides to give,

$$q^2 = \frac{x - p}{2}$$

Then multiply by 2 gives,

$$2q^2 = x - p$$

and then adding p to both sides gives

$$2q^2 + p = x$$

or

$$x = 2q^2 + p.$$

Example 5

Make c the subject of the formula,

$$x = b + \sqrt{c - q}.$$

Solution

It is not possible to remove the square root unless it is the whole of one side of the equation of formula that lies within the square root. So the first step is to subtract b from both sides to give,

$$x - b = \sqrt{c - q}$$

Now both sides can be squared to remove the square root and give,

$$(x - b)^2 = c - q$$

Finally q can be added to both sides to give,

$$(x - b)^2 + q = c$$

or

$$c = (x - b)^2 + q.$$

Example 6

Make x the subject of the formula

$$p = x^2 - 6$$

Solution

First add 6 to both sides to give,

$$p + 6 = x^2$$

Now taking the square root of both sides gives,

$$\pm\sqrt{p+6} = x$$

or

$$x = \pm\sqrt{p+6}$$

Note that when a square root is taken it is important to insert the ± sign.

Example 7

Make x the subject of the formula,

$$z = (x + 7)^2 - y$$

Solution

First add y to both sides to give,

$$z + y = (x + 7)^2.$$

Then take the square root and include the ± sign, to give

$$\pm\sqrt{z+y} = x + 7$$

Finally subtract 7 from each side to give,

$$\pm\sqrt{z+y} - 7 = x$$

or

$$x = -7 \pm \sqrt{z+y}.$$

Exercises

1. Make x the subject of each formula below:

(a) $p = q - \dfrac{1}{x}$

(b) $c = \dfrac{q-5}{x}$

(c) $\dfrac{1}{c} = \dfrac{4}{x} + \dfrac{2}{z}$

(d) $Q = \dfrac{p}{x} - \dfrac{q}{x}$

(e) $y = \dfrac{1}{x+2}$

(f) $y = \dfrac{2x+z}{x}$

2. Make x the subject of each formula below.

(a) $y = \sqrt{x-5}$

(b) $y = 2 + \sqrt{10-x}$

(c) $q = 5 + \sqrt{4x-2}$

(d) $p = (x-7)^2$

(e) $q = (3+x)^2 + 2$

(f) $z = \dfrac{2}{x^2}$

3. The distance, d, fallen in t seconds is given by

$$d = \frac{gt^2}{2}$$

Find an expression for the time it takes to fall a distance d.

4. (a) When two resistors are connected in parallel, their combined resistance can be found using,

$$\frac{1}{R} = \frac{1}{R_A} + \frac{1}{R_B}.$$

Make R_A the subject of the expression.

(b) For three resistors this becomes,

$$\frac{1}{R} = \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}$$

Make R_A the subject of this expression.

Solutions

1. (a) $p = q - \dfrac{1}{x}$

$px = qx - 1$

$px - qx = -1$

$x(p - q) = -1$

$x = \dfrac{-1}{p-q}$

$= \dfrac{1}{q-p}$

(d) $Q = \dfrac{p}{x} - \dfrac{q}{x}$

$Qx = p - q$

$x = \dfrac{p-q}{Q}$

(b) $C = \frac{q-5}{x}$

$Cx = q - 5$

$x = \frac{q-5}{C}$

(c) $\frac{1}{c} = \frac{4}{x} + \frac{2}{z}$

$\frac{cxz}{c} = \frac{4cxz}{x} + \frac{2cxz}{z}$

$xz = 4cz + 2cx$

$xz - 2cx = 4cz$

$x(z - 2c) = 4cz$

$x = \frac{4cz}{z-2c}$

(e) $y = \frac{1}{x+2}$

$y(x + 2) = 1$

$x + 2 = \frac{1}{y}$

$x = \frac{1}{y} - 2$

(f) $y = \frac{2x+z}{x}$

$xy = 2x + z$

$xy - 2x = z$

$x(y - 2) = z$

$x = \frac{z}{y-2}$

2. (a) $y = \sqrt{x-5}$

$y^2 = x - 5$

$y^2 + 5 = x$

$x = y^2 + 5$

(b) $y = 2 + \sqrt{10-x}$

$y - 2 = \sqrt{10-x}$

$(y - 2)^2 = 10 - x$

$(y - 2)^2 + x = 10$

$x = 10 - (y - 2)^2$

(d) $p = (x - 7)^2$

$\pm\sqrt{p} = x - 7$

$\pm\sqrt{p} + 7 = x$

$x = 7 \pm\sqrt{p}$

(e) $q = (3 + x)^2 + 2$

$q - 2 = (3 + x)^2$

$\pm\sqrt{q-2} = 3 + x$

$\pm\sqrt{q-2} - 3 = x$

$x = -3 \pm\sqrt{q-2}$.

(c) $q = 5 + \sqrt{4x-2}$

$q - 5 = \sqrt{4x-2}$

$(q-5)^2 = 4x - 2$

$(q-5)^2 + 2 = 4x$

$\dfrac{(q-5)^2+2}{4} = x$

.

$x = \dfrac{(q-5)^2+2}{4}$

(f) $z = \dfrac{2}{x^2}$

$x^2 z = 2$

$x^2 = \dfrac{2}{z}$

$x = \pm\sqrt{\dfrac{2}{z}}$

3. $d = \dfrac{gt^2}{2}$

$2d = gt^2$

$\dfrac{2d}{g} = t^2$

$t = \pm\sqrt{\dfrac{2d}{g}}$

4. (a) $\dfrac{1}{R} = \dfrac{1}{R_A} + \dfrac{1}{R_B}$

$\dfrac{RR_A R_B}{R} = \dfrac{RR_A R_B}{R_A} + \dfrac{RR_A R_B}{R_B}$

$R_A R_B = RR_B + RR_A$

$R_A R_B - RR_A = RR_B$

$R_A (R_B - R) = RR_B$

$R_A = \dfrac{RR_B}{R_B - R}$

(b) $\dfrac{RR_A R_B R_C}{R} = \dfrac{RR_A R_B R_C}{R_A} + \dfrac{RR_A R_B R_C}{R_B} + \dfrac{RR_A R_B R_C}{R_C}$

$R_A R_B R_C = RR_B R_C + RR_A R_C + RR_A R_B$

$R_A R_B R_C - RR_A R_C - RR_A R_B = RR_B R_C$

$R_A (R_B R_C - RR_C - RR_B) = RR_B R_C$

$R_A = \dfrac{RR_B R_C}{R_B R_C - RR_C - RR_B}$

MATHEMATICS SUPPORT SERIES

Basics of Algebra 4

Indices and Powers

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Introduction

This unit is concerned with the use of indices and powers. Powers and indices occur in very many areas. Some examples include financial applications, such as loans, compound interest and mortgage repayments. In the life science models for growth and decay often involve powers. To be able to understand the meaning of the powers that you encounter and to work efectively with them, it is important that you are aware of the notation used and their rates for manipulating powers and indices. This booklet will help you to:

(i) understand the meanings of indices or powers,

(ii) manipulate powers and indices.

Powers and Indices

This section explores the basic ideas of powers and defines what is understood by powers.

Powers that are Positive Whole Numbers

When a number or letter has a power that is a positive whole number it simply means that the number is multiplied by about as many times as the power indicates.

Examples

$7^2 = 7 \times 7 = 49$

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

$4^1 = 4$

$6^3 = 6 \times 6 \times 6 = 216$

$a^3 = a \times a \times a$

$b^7 = b \times b \times b \times b \times b \times b \times b$

$x^1 = x$

Exercises

1. Find

(a)	2^4	(d)	10^4
(b)	3^2	(e)	8^3
(c)	7^4	(f)	6^4

2. Write each expression below as a single letter to a power.

(a)	$a \times a \times a \times a$	(d)	$x \times x \times x \times x \times x \times x$
(b)	$b \times b$	(e)	$b \times b \times b \times b \times b$
(c)	q	(f)	$p \times p \times p$

Solutions

1. (a) $2^4 = 2 \times 2 \times 2 \times 2$
$= 16$

(b) $3^2 = 3 \times 3$
$= 9$

(c) $7^4 = 7 \times 7 \times 7 \times 7 \times 7$
$= 2401$

(d) $10^4 = 10 \times 10 \times 10 \times 10$
$= 10000$

(e) $8^3 = 8 \times 8 \times 8$
$= 512$

(f) $6^4 = 6 \times 6 \times 6 \times 6$
$= 1296$

2. (a) a^4

(b) b^2

(c) q^1

(d) x^6

(e) b^5

(f) p^3

Powers that are Negative Whole Numbers

When a power is negative, then this describes a fraction which has 1 on the top but underneath is the number to the same power with its sign changed.

Examples

1. $$2^{-4} = \frac{1}{2^4} = \frac{1}{2 \times 2 \times 2 \times 2} = \frac{1}{16}.$$

2. $$9^{-1} = \frac{1}{9}$$

3. $$4^{-2} = \frac{1}{4^2} = \frac{1}{4 \times 4} = \frac{1}{16}$$

4. $a^{-4} = \dfrac{1}{a^4}$

$= \dfrac{1}{a \times a \times a \times a}$

5. $x^{-2} = \dfrac{1}{x^2}$

$= \dfrac{1}{x \times x}$

Exercises

1. Write each expression as a fraction containing no powers.

(a) 10^{-1} (d) 6^{-2}

(b) 3^{-3} (e) 5^{-3}

(c) 2^{-5} (f) 10^{-2}

2. Write each expression as a letter to a single power.

(a) $\dfrac{1}{a \times a \times a \times a \times a}$ (c) $\dfrac{1}{b \times b \times b \times b}$

(b) $\dfrac{1}{a \times a \times a}$ (d) $\dfrac{1}{x \times x \times x \times x \times x \times x}$

Solutions

1. (a) $10^{-1} = \dfrac{1}{10}$ (d) $6^{-2} = \dfrac{1}{6^2} = \dfrac{1}{36}$

(b) $3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$ (e) $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$

(c) $2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32}$ (f) $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$

2. (a) $\dfrac{1}{a \times a \times a \times a \times a} = \dfrac{1}{a^5} = a^{-5}$ (c) $\dfrac{1}{b \times b \times b \times b} = \dfrac{1}{b^4} = b^{-4}$

(b) $\dfrac{1}{a \times a \times a} = \dfrac{1}{a^3} = a^{-3}$ (d) $\dfrac{1}{x \times x \times x \times x \times x \times x} = \dfrac{1}{x^6} = x^{-6}$

Powers that are Fractions

When powers are of the form $\frac{1}{n}$, they indicate the nth root of a number. For example $\frac{1}{2}$ indicates the square roots, while $\frac{1}{3}$ indicates a cube root.

Examples

1. $9^{\frac{1}{2}} = \sqrt{9} = 3$

2. $16^{\frac{1}{2}} = \sqrt{16} = 4$

3. $25^{\frac{1}{2}} = \sqrt{25} = 5$

4. $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

5. $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

6. $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

When a power has a number other than 1 on the top then a root of the number has to be found and raised to another power. For example $9^{\frac{3}{2}}$ requres the square root of 9 to be cubed. Alternatively it is possible to find the square root of 9^3. Check that you can obtain the same result of 27 by either approach.

Examples

1. $16^{\frac{5}{2}} = \left(\sqrt{16}\right)^5$

   $= 4^5$

   $= 1024$

2. $9^{\frac{7}{2}} = \left(\sqrt{9}\right)^7$

   $= 3^7$

   $= 2187$

3. $8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4$

   $= 2^4$

   $= 16$

4 $(100)^{\frac{3}{2}} = \left(\sqrt{100}\right)^3$

$= 10^3$

$= 1000$

Note that the usual convention is to write the numbers as top heavy fractions such as $\frac{7}{2}$ rather than the equivalent mixed number 3½.

Exercises

1. Find

(a) $4^{\frac{3}{2}}$

(b) $25^{\frac{3}{2}}$

(c) $64^{\frac{3}{2}}$

(d) $125^{\frac{2}{3}}$

(e) $1000^{\frac{2}{3}}$

(f) $16^{\frac{5}{4}}$

(g) $9^{\frac{5}{2}}$

(h) $49^{\frac{5}{2}}$

Solutions

1. (a) $4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

(b) $25^{\frac{3}{2}} = \left(\sqrt{25}\right)^3 = 5^3 = 125$

(c) $64^{\frac{3}{2}} = \left(\sqrt{64}\right)^3 = 8^3 = 512$

(d) $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$

(e) $1000^{\frac{2}{3}} = \left(\sqrt[3]{1000}\right)^2 = 10^2 = 100$

(f) $16^{\frac{5}{4}} = \left(\sqrt[4]{16}\right)^5 = 2^5 = 32$

(g) $9^{\frac{5}{2}} = \left(\sqrt{9}\right)^5 = 3^5 = 243$

(h) $49^{\frac{5}{2}} = \left(\sqrt{49}\right)^5 = 7^5 = 16807$

Rules of Indices

There are some basic rules for manipulating algebraic terms that involve powers. These are now explored.

Rules for Multiplication

Consider

$$x^3 \times x^2 = (x \times x \times x) \times (x \times x)$$
$$= x^5$$

Note that when x^3 is multiplied by x^2 the result is x^5, the powers have been added together. Adding the powers is true in any situation where the same letter or number appears in both terms that are to be multiplied together.

As a second example consider,

$$2^4 \times 2^5 = 16 \times 32$$
$$= 512$$
$$\underline{\underline{= 2^9}}$$

In general

$$a^m \times a^n = a^{m+n}$$

Examples

1. $2^4 \times 2^3 = 2^{4+3}$

   $= 2^7$

2. $a^5 \times a^3 = a^{5+3}$

   $= a^8$

3. $z^2 \times z^7 = z^{2+7}$

   $= z^9$

4. $z^4 \times z^{½} = z^{4½}$

   $= z^{9/2}$

5. $x^6 \times x^{-4} = x^{6+(-4)}$

   $= x^2$

6. $x^2 \times y^2$ This cannot be simplified because no letters are repeated.

Exercises

1. For the example below state which can and cannot be simplified. Simplify those for which it is possible

(a) $2^6 \times 2^7$

(b) $3^4 \times 3^5$

(c) $4^2 \times 5^6$

(d) $3^7 \times 3^{-8}$

(e) $4^5 \times 4^{-2}$

(f) $a^2 \times b^2$

(g) $a^4 \times a^7$

(h) $b^6 \times b^{-2}$

(i) $p^m \times p^n$

(j) $a^4 \times a^8$

(k) $a^5 \times a^{\frac{1}{2}}$

(l) $x^5 \times x^{-2}$

(m) $x^{\frac{1}{2}} \times x^{\frac{1}{4}}$

(n) $x^7 \times x^{-13}$

(o) $x^{½} \times x^{-1}$

Solutions

1. (a) $2^6 \times 2^7 = 2^{6+7} = 2^{13}$

(b) $3^3 \times 3^5 = 3^{4+5} = 3^9$

(c) $4^2 \times 5^6$ Impossible because different numbers involved.

(d) $3^7 \times 3^{-8} = 3^{7+(-8)} = 3^{-1} = \dfrac{1}{3}$

(e) $4^5 \times 4^{-2} = 4^{5+(-2)} = 4^3$

(f) $a^2 \times b^2$ Impossible because different letters involved.

(g) $a^4 \times a^7 = a^{4+7} = a^{11}$

(h) $b^6 \times b^{-2} = b^{6+(-2)} = b^4$

(i) $p^m \times p^n = p^{m+n}$

(j) $a^4 \times a^8 = a^{4+8} = a^{12}$

(k) $a^5 \times a^{\frac{1}{2}} = a^{5+\frac{1}{2}} = \mathrm{a}^{\frac{11}{2}}$

(l) $x^5 \times x^{-2} = x^{5+(-2)} = x^3$

(m) $x^{\frac{1}{2}} \times x^{\frac{1}{4}} = x^{\frac{1}{2}+\frac{1}{4}} = x^{\frac{3}{4}}$

(n) $x^7 \times x^{-13} = x^{7+(-13)} = x^{-6}$

(o) $x^{\frac{1}{2}+(-1)} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

Rules for Division

Consider

$$\frac{2^5}{2^3} = \frac{32}{8}$$

$$= 4$$

$$= 2^2.$$

Notice that final power of 2 is given by 5 – 3, the difference between the original powers.

A second example shows the same process,

$$\frac{5^6}{5^3} = \frac{15625}{125}$$

$$= 125$$

$$= 5^3.$$

Here again the final power is the difference between the two original powers.

In general

$$\frac{a^m}{a^n} = a^{m-n}$$

Examples

1. $$\frac{2^6}{2^4} = 2^{6-4}$$

   $$\underline{\underline{= 2^2}}$$

2. $$\frac{7^4}{7^8} = 7^{4-8}$$

   $$= 7^{-4}$$

   $$\underline{\underline{= \frac{1}{7^4}}}$$

3. $\dfrac{4^6}{4^{\frac{1}{2}}} = 4^{6-\frac{1}{2}}$

$= 4^{5\frac{1}{2}}$

$= 4^{\frac{11}{2}}$

4. $\dfrac{a^6}{a^2} = a^{6-2}$

$= a^4$

5. $\dfrac{a^3}{a^7} = a^{3-7}$

$= a^{-4}$

$= \dfrac{1}{a^4}$

An important result that concerns x^0 can now be established. A value for x^0 can be determined by stating,

$\dfrac{x^n}{x^n} = x^{n-n}$

$= x^0.$

But $\dfrac{x^n}{x^n} = 1$ so for any value of x, $x^0 = 1$.

Exercises

1. Find

(a) 3^0 (c) 0^4

(b) 4^0 (d) a^0

2. Simplify where possible the expression below.

(a)	$\frac{2^8}{2^4}$	(e)	$\frac{x^2}{x^4}$	(i)	$\frac{x^7}{x^7}$
(b)	$\frac{3^6}{3^8}$	(f)	$\frac{x^6}{y^2}$	(j)	$\frac{x^8}{x^{-1}}$
(c)	$\frac{5^{10}}{5^9}$	(g)	$\frac{p^7}{p^{1/2}}$	(k)	$\frac{x^{-6}}{x^{-2}}$
(d)	$\frac{6^2}{6^3}$	(h)	$\frac{p^{3/2}}{p^{1/2}}$	(l)	$\frac{x^{-5}}{x^5}$

Solutions

1. (a) $3^0 = 1$ (c) $0^4 = 0$

   (b) $4^0 = 1$ (d) $a^0 = 1$

2. (a) $\frac{2^8}{2^4} = 2^{8-4} = 2^4$

   (b) $\frac{3^6}{3^8} = 3^{6-8} = 3^{-2} = \frac{1}{3^2}$

   (c) $\frac{5^{10}}{5^9} = 5^{10-9} = 5^1 = 5$

   (d) $\frac{6^2}{6^3} = 6^{2-3} = 6^{-1} = \frac{1}{6}$

   (e) $\frac{x^2}{x^4} = x^{2-4} = x^{-2} = \frac{1}{x^2}$

   (f) $\frac{x^6}{y^2}$ Impossible because different letters

   (g) $\frac{p^7}{p^{1/2}} = p^{7-½} = p^{6½} = p^{13/2}$

   (h) $\frac{p^{3/2}}{p^{1/2}} = p^{3/2-1/2} = p^1 = p$

   (i) $\frac{x^7}{x^7} = x^{7-7} = x^o = 1$

   (j) $\frac{x^8}{x^{-1}} = x^{8-(-1)} = x^9$

   (k) $\frac{x^{-6}}{x^{-2}} = x^{-6-(-2)} = x^{-4}$

   (l) $\frac{x^{-5}}{x^5} = x^{-5-5} = x^{-10}$

A Power to a Power

Sometimes a situation arises where a letter or number to a power is raised to another power. For example,

$$(2^2)^3 = 4^3$$

$$= 64$$

$$= 2^6$$

Here is a situation where the original powers have been multiplied together to give the final power.

This is true in general and can be stated as,

$$\left(a^m\right)^n = a^{m\times n}$$

Examples

1. $(3^2)^6 = 3^{2\times 6}$

   $= 3^{12}$

2. $(5^3)^6 = 5^{3\times 6}$

   $= 5^{18}$

3. $\sqrt{(4^{18})} = (4^{18})^{\frac{1}{2}}$

   $= 4^{18\times \frac{1}{2}}$

   $= 4^9$

4. $\sqrt[3]{(5^6)} = (5^6)^{\frac{1}{3}}$

   $= 5^{6\times \frac{1}{3}}$

   $= 5^2$

5. $(a^4)^{-2} = a^{4\times (-2)}$

   $= a^{-8}$

   $= \frac{1}{a^8}$

6. $(\sqrt{3})^4 = (3^{\frac{1}{2}})^4$

   $= 3^{\frac{1}{2}\times 4}$

   $= 3^2$

Exercises

1. Simplify each expression below.

(a)	$(2^2)^8$	(f)	$(\sqrt{5})^{-4}$	(k)	$(2x^2)^3$
(b)	$(3^4)^5$	(g)	$(\sqrt[3]{6})^8$	(l)	$\left(\frac{1}{x^3}\right)^5$
(c)	$(4^{-1})^4$	(h)	$\sqrt[3]{(x^2)}$	(m)	$(p^2q^3)^4$
(d)	$(5^{-2})^{-3}$	(i)	$\sqrt{\frac{1}{x}}$	(n)	$\left(\frac{a^4}{b^8}\right)^{\frac{1}{2}}$
(e)	$(6^{-1})^{-1}$	(j)	$(\sqrt{x})^{-2}$	(o)	$(\sqrt{x})^{-10}$

Solutions

1. (a) $(2^2)^8 = 2^{2\times 8} = 2^{16}$

(b) $(3^4)^5 = 3^{4\times 5} = 3^{20}$

(c) $(4^{-1})^4 = 4^{(-1)\times 4} = 4^{-4} = \dfrac{1}{4^4}$

(d) $(5^{-2})^{-3} = 5^{(-2)\times(-3)} = 5^6$

(e) $(6^{-1})^{-1} = 6^{(-1)\times(-1)} = 6^1 = 6$

(f) $(\sqrt{5})^{-4} = \left(5^{\frac{1}{2}}\right)^{-4} = 5^{\frac{1}{2}\times(-4)} = 5^{-2} = \dfrac{1}{5^2}$

(g) $(\sqrt[3]{6})^8 = \left(6^{\frac{1}{3}}\right)^8 = 6^{\frac{1}{3}\times 8} = 6^{\frac{8}{3}}$

(h) $\sqrt[3]{(x^2)} = (x^2)^{\frac{1}{3}} = x^{2\times\frac{1}{3}} = x^{\frac{2}{3}}$

(i) $\sqrt{\dfrac{1}{x}} = (x^{-1})^{\frac{1}{2}} = x^{(-1)\times\frac{1}{2}} = x^{-\frac{1}{2}}$

(j) $(\sqrt{x})^{-2} = \left(x^{\frac{1}{2}}\right)^{-2} = x^{\frac{1}{2}\times(-2)} = x^{-1} = \dfrac{1}{x}$

(k) $(2x^2)^3 = 2^3 \times x^{2\times 3} = 8x^6$

(l) $\left(\dfrac{1}{x^3}\right)^5 = (x^{-3})^5 = x^{-3\times 5} = x^{-15}$

(m) $(p^2q^3)^4 = p^{2\times 4}\, q^{3\times 4} = p^8\, q^{12}$

(n) $\left(\dfrac{a^4}{b^8}\right)^{\frac{1}{2}} = \dfrac{a^{4\times\frac{1}{2}}}{b^{8\times\frac{1}{2}}} = \dfrac{a^2}{b^4}$

(o) $(\sqrt{x})^{10} = \left(x^{\frac{1}{2}}\right)^{10} = x^{\frac{1}{2}\times 10} = x^5$

Mathematics Support Series

Straight Line Graphs

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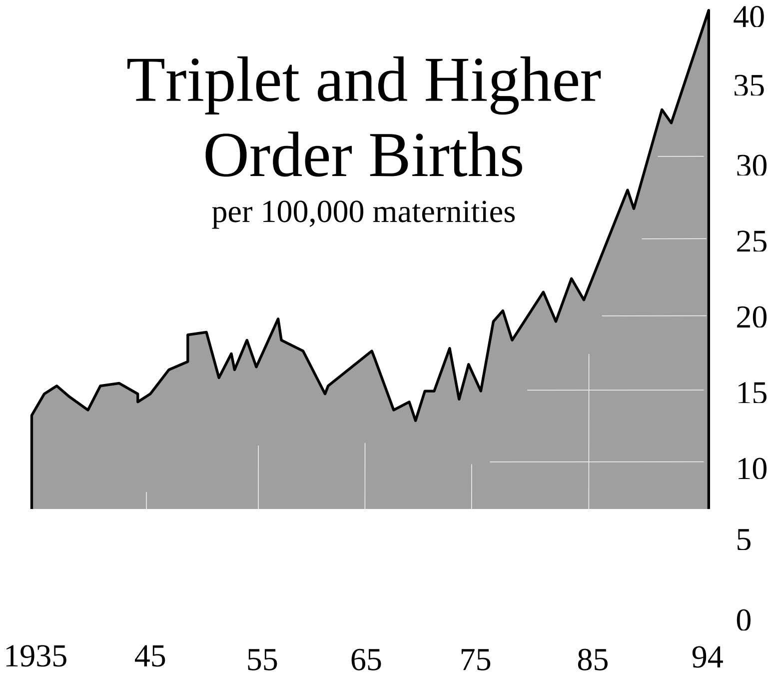
Introduction

Graphs are often used as a tool to display the results of experimental work or to display data of different kinds. The graph below was taken from a daily paper. The trend that it shows are complete and the paper claims that the sharp increase is due to the use of fertility treatments on an increasing scale.



Not all graphs are as complex as this one and this booklet will concentrate on straight line graphs. This type of graph can occur in a variety of situations and can be used as an approximation when data points do not lie exactly on a straight line.

Graphs can be very useful for solving problems or making predictions. This unit will help you to;

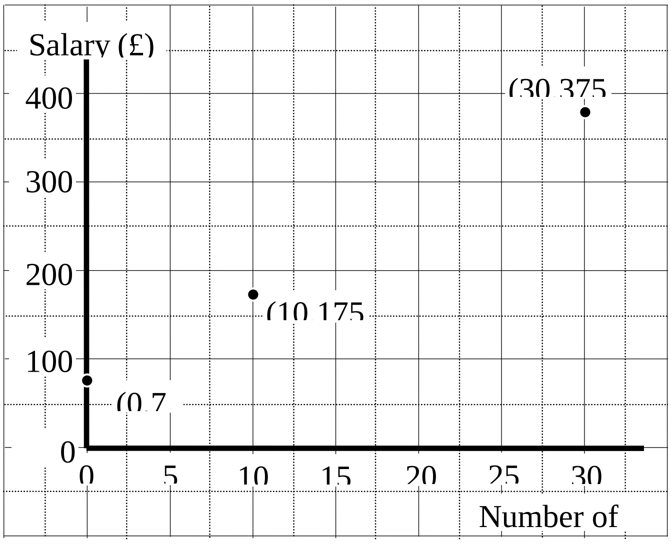
(i) draw a graph,
(ii) use a graph to make predictions,
(iii) find the gradient of a straight line graph,
(iv) find the equation of a straight line graph.

Plotting Graphs From Given Data

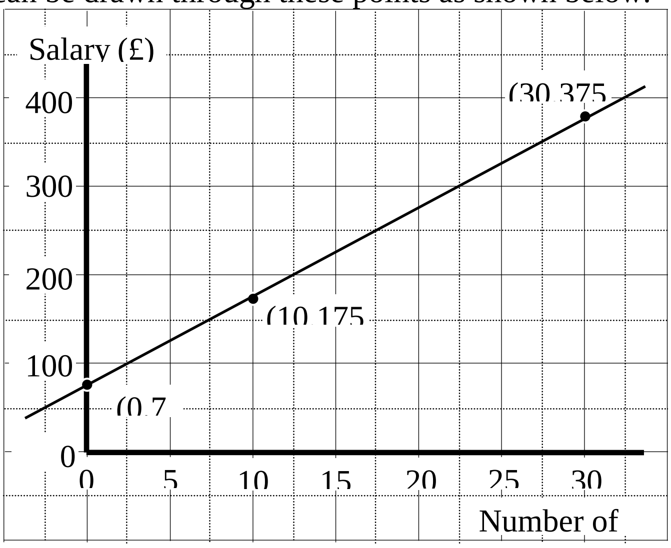
Example 1

A salesman earns a minimum of £75 per week, even if he makes no sales. If he makes 10 sales he earns £175, and if he makes 30 sales, £375.

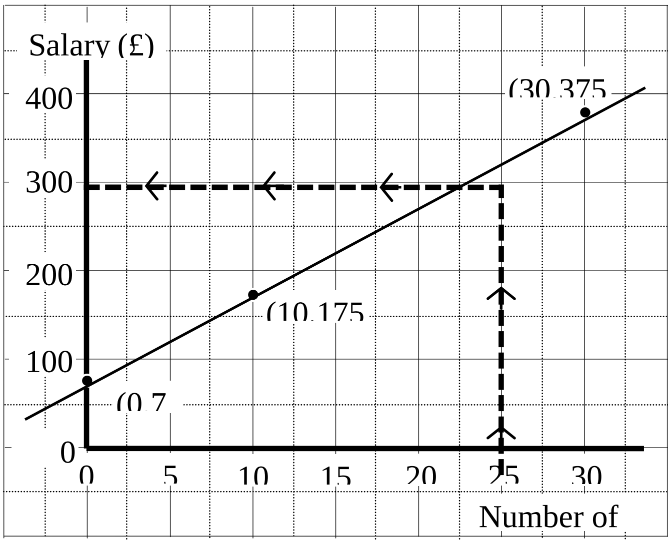
There is a relationship between the sales of the salesman and the number of sales. This can be represented on a graph. The vertical scale could be labelled, Sales, and the horizontal, Number of Sales. Often the quantities are represented by single letters, in this case *S* for salary and *n* for number of sales.



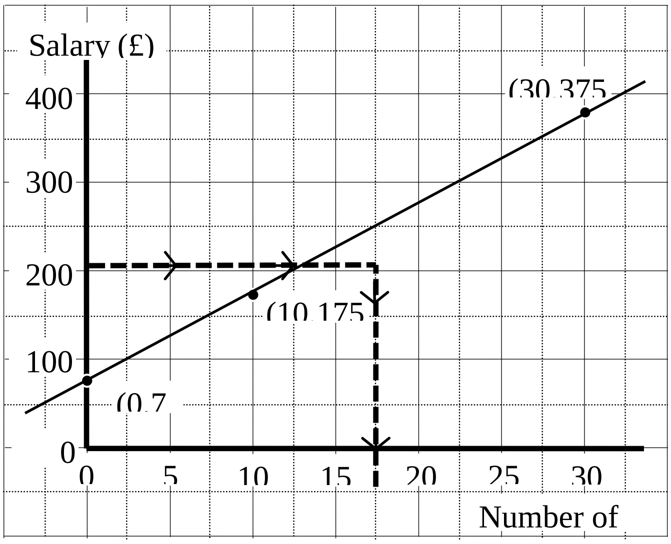
The graph shown above has 3 points marked in, each corresponding to one of the 3 cases given at the beginning of this example. The numbers in brackets next to each point are known as the coordinates of that point, for example (10,175), means that n=10 and S=175. Note the value for the horizontal scale appears first.
A straight line can be drawn through these points as shown below.



The graph can be used to make predictions. To find the salary for 25 sales, begin at 25 on the horizontal axis, draw a line straight up until it meets the graph. Then draw a horizontal line that reaches the vertical axis, to obtain a salary of £325. This is shown below.



Also consider a salesman who wishes to earn £250 per week. How many sales must he make. Start on the graph at £250 and draw a horizontal line to the graph, then a vertical line down. This gives 17.5 sales, so to earn at least £250, 18 sales are in fact required. This process is shown below.



Problems

1. Use the graph to find the weekly salary for;

   (a) 8 (b) 12 (c) 20 sales.

2. Use the graph to find the number of sales required for a weekly salary of at least

   (a) £225 (b) £275 (c) £150

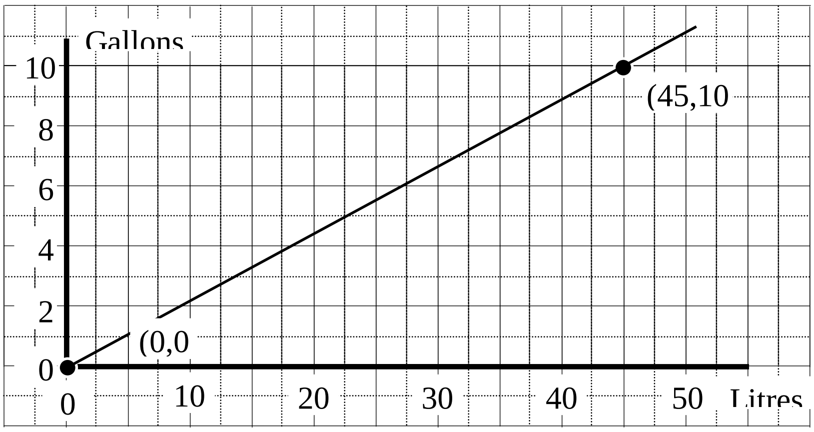
Answers

1. (a) £155 (b) £195 (c) £275

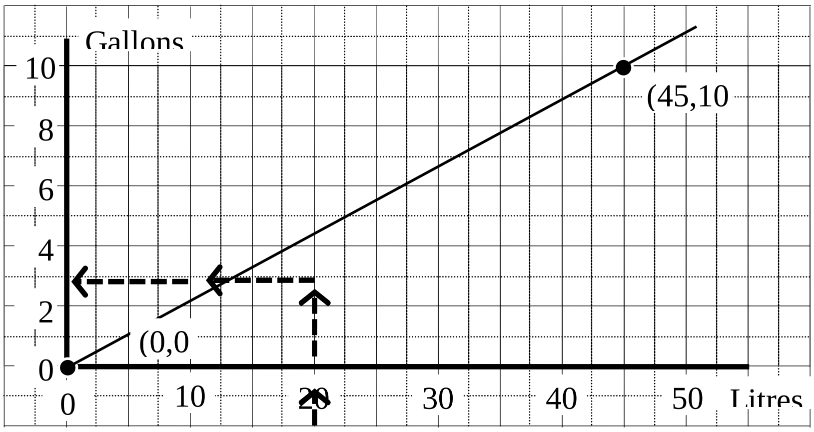
2. (a) 15 (b) 20 (c) 8

Example 2

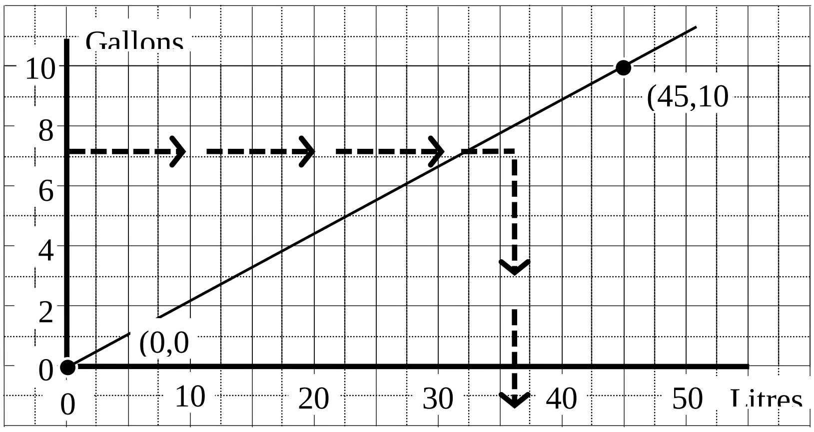
In this example a graph will be used to convert gallons to litres. The information that is available is that 10 gallons is equivalent to 45 litres. The graph below shows 2 points (45,10) which corresponds to the information given and (0,0) as zero is the same using both units. A line has been drawn through the two points.



The graph below shows how to convert 20 litres into gallons. What is the value obtained in gallons?



The graph below shows how to convert 8 gallons into litres. What is the value obtained in litres?



Problems

1. Convert each volume below into litres;

(a) 7 gallons (b) 1 gallon (c) 5 gallons.

2. Convert each volume below in litres into gallons;

(a) 30 litres (b) 5 litres (c) 40 litres.

Answers

1.	(a)	31.5 litres	(b)	4.5 litres	(c)	22.5 litres.
2.	(a)	6.7 gallons	(b)	1.1 gallons	(c)	8.9 gallons.

Exercises

1. The cost of manufacturing 100 cameras is £700 and the cost of manufacturing 150 cameras is £950.

(i) Draw a graph with cost on the vertical axis and number of cameras on the horizontal. Plot the 2 points that represent the information given above and draw a straight line through them.

(ii) Use you graph to find the cost of manufacturing;

(a) 50 cameras (b) 125 cameras.

(iii) If £250 is available, how many cameras can be made?

2. It is known that 200 kilometres is equivalent to 124 miles.

(i) Draw a conversion graph.

(ii) Convert the distances given below to kilometres;

(a) 50 miles (b) 100 miles

(iii) Convert the distances given below to miles;

(a) 10 km (b) 100 km

3. A salesman earns £120 in a week when he sells 4 items and £250 when selling 30 items.

(i) Plot a graph of earnings on vertical against number of sales on the horizontal, assuming that the graph is a straight line passing through the points that represent the information given above.

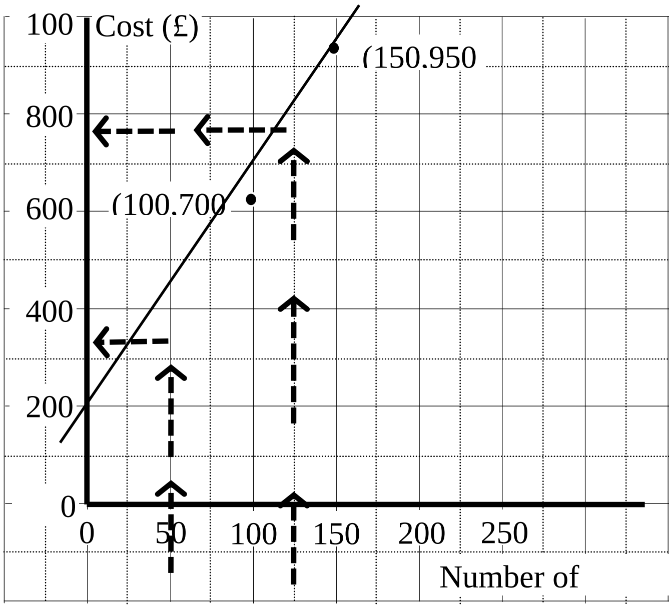
(ii) Find the minimum earnings for the salesman.

(iii) What would be the earnings for 20 sales?

(iv) How many items must be sold to earn £180?

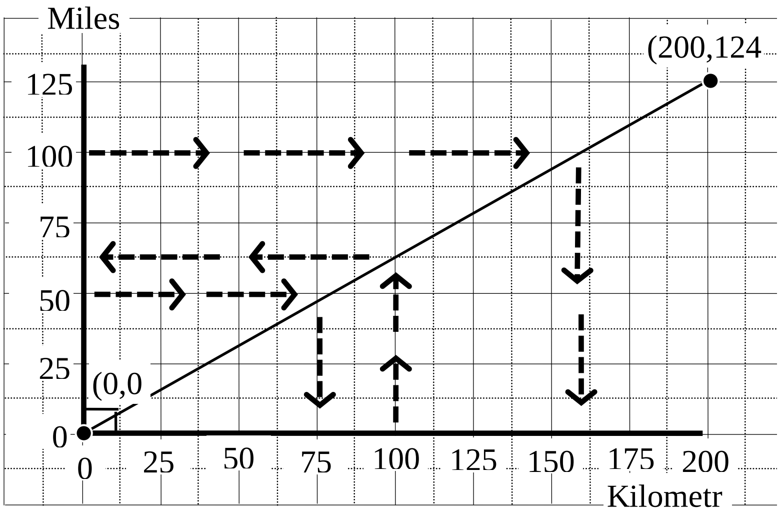
Solutions

1.



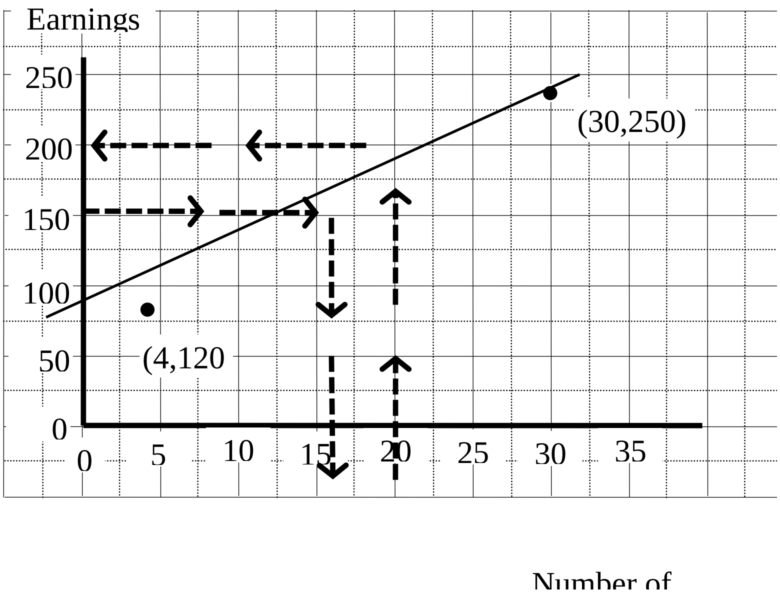
(ii) (a)£450 (b) £825.
(iii) 10 cameras.

2.



(ii) (a) 81 km (b) 161 km .
(iii) (a) 6 miles (b) 62 miles.

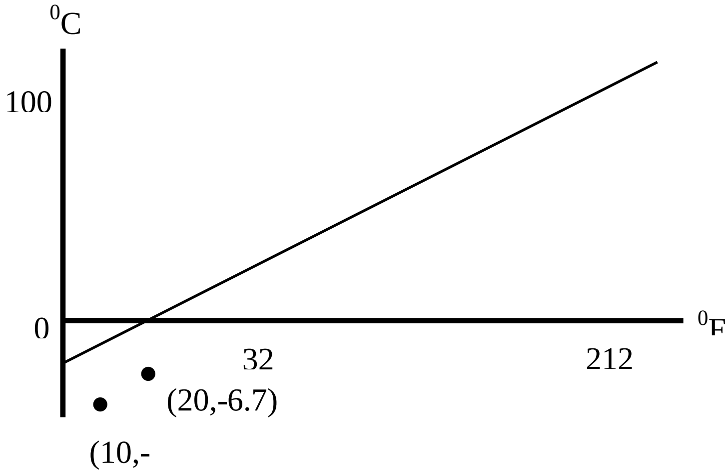
3.



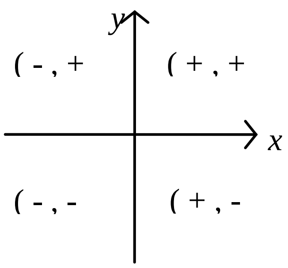
(ii) £100 (iii) £200 (iv) 16 items.

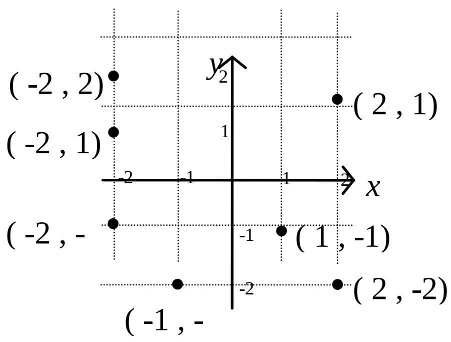
Negative Coordinates

So far the graphs that have been considered have involved coordinates where both numbers have been positive, but in many instances negative numbers will also appear. For example the graph below can be used for converting between temperatures in Fahrenheit and centigrade.



Note that the coordinates of the two points marked contain a negative number, because they are below the horizontal axis. If points are positioned to the left of the vertical axis, then the first number of each pair of coordinates will be negative. The diagram on the left below shows how in each of the four quadrants the coordinates take positive or negative values.

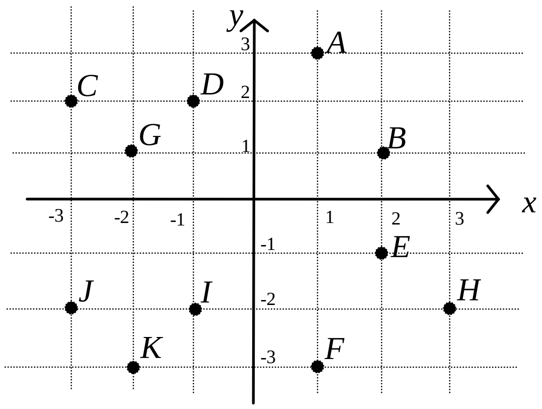




The diagram on the right above shows a number of points and their coordinates.

Exercise

1. Write down the coordinates of each point marked on the axes below.



2. On a set of axes plot and label each of the following points.

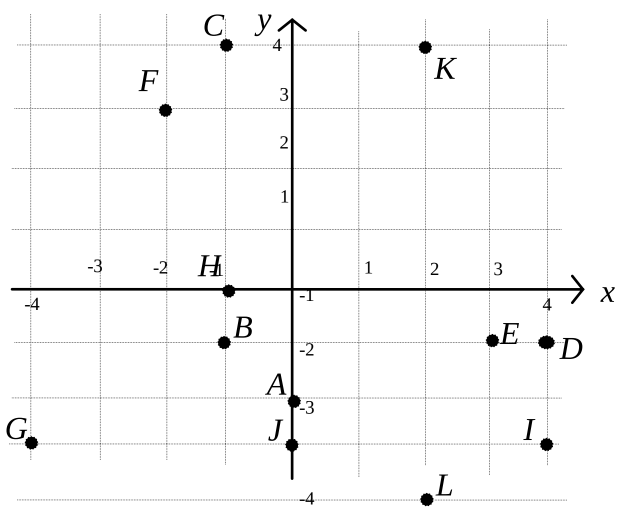
A	(0, –2)	E	(3, –1)	I	(4, –3)
B	(–1, –1)	F	(–2, 3)	J	(0, –3)
C	(–1, 4)	G	(–4, –3)	K	(2, 4)
D	(4, –1)	H	(–1, 0)	L	(2, –4)

Solutions

1.

A	(1, 3)	D	(–1, 2)	G	(–2, 1)	J	(–3, –2)
B	(2, 1)	E	(2, –1)	H	(3, –2)	K	(–2, –3)
C	(–3, 2)	F	(1, –3)	I	(–1, –2)		

2.



Graphs From Equations

Often an equation that defines the relationship between two qualities will be given and this then has to be used to find points that allow a graph to be drawn.

Example 1

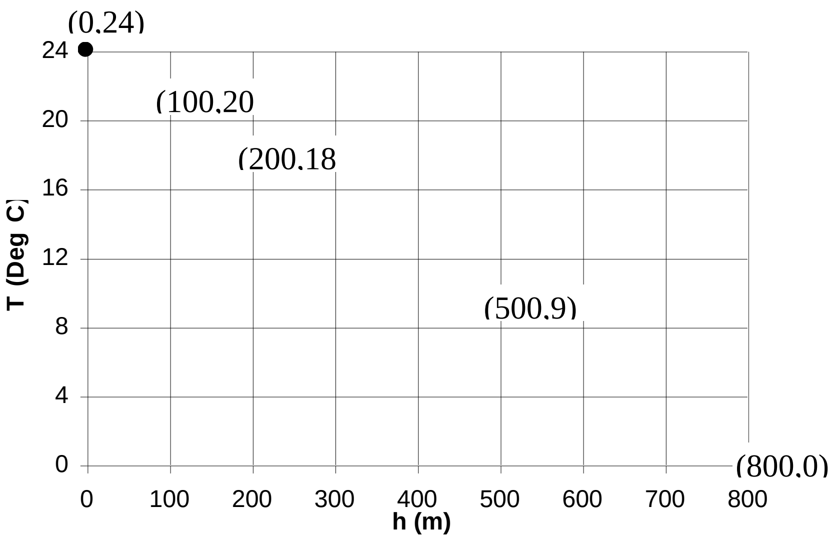
The air temperature decreases as you climb a mountain. On a particular day the temperature, *T*, in degrees Centigrade is related to the height climbed, *h*, in miles. The relationship is such that;

$$T = 24 - 0.03h$$

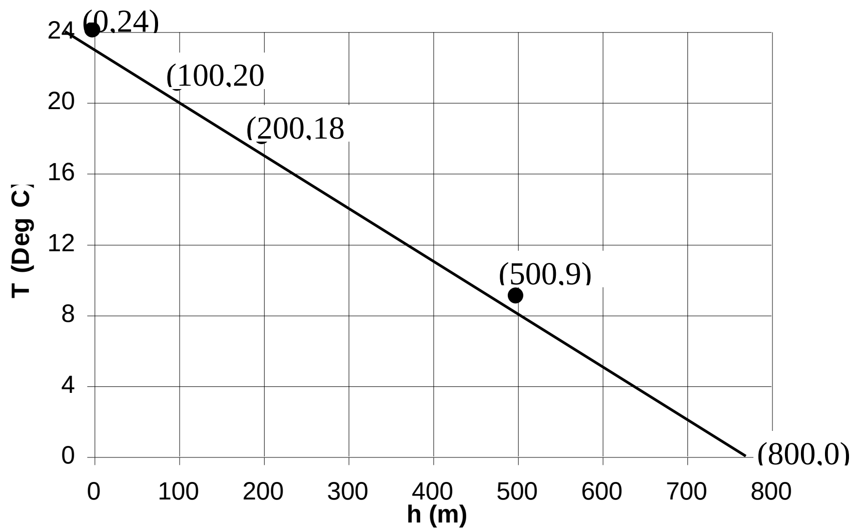
To plot a graph of temperature against height it is necessary to calculate the temperature for some different heights. The table below has been partially completed. Calculate the values that should be placed in the empty boxes.

h (m)	0	100	200	500	800
T (°C)	24°	21°		9°	

These values can than be plotted on a graph as shown below.



A Straight line can then be drawn through these points as below.



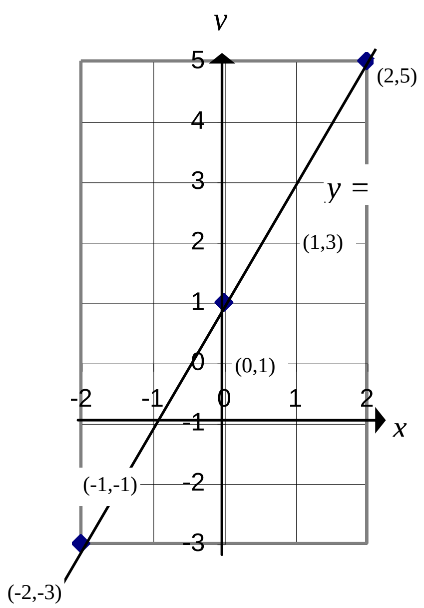
Example 2

Sometimes graphs are considered that do not relate directly to a context. In this case the quantities that are normally used are *y* on the vertical and *x* on the horizontal. Also it is sometimes necessary to include negative values of the quantities being considered.

To draw the graph that has the equation $y = 2x + 1$ it is necessary to calculate a table of values as in the previous example. The table below has been partially completed, find the missing values.

x	-2	-1	0	1	2
y	-3		1		5

These points can now be plotted on a graph.
A straight line can then be drawn through the points to give the line shown.



Exercise

1. The cost, C in pence, of processing x kilograms of coffee beans is given by;

$$C = 300 + 0.5x.$$

(i) Complete the table below;

x	0	200	400	600	800	1000
C			500			

(ii) Plot the points on a set of axes and verify that the points lie on a straight line.

(iii) Use the graph to find how many kilograms of coffee can be processed for £7.50.

2. The value of a machine decrease with time according to the equation;

$$V = 1500 - 120t$$

where V is the value in £ and t the time in years.

(i) Complete the table below:

t	0	2	4	6	8	10	12
V							

(ii) Plot the points on a graph and draw a straight line through them.

(iii) When does the value of the machine become zero?

(iv) When is the value of the machine £1000?

3. Draw the graph with equation $y = 3x - 2$ by first completing the table below.

x	-2	-1	0	1	2
y	-8				4

4. By completing a copy of the table below, draw the graph for each question.

x	-3	-2	-1	0	1	2	3
y							

(i) $y = 2x - 1$ (ii) $y = x + 2$

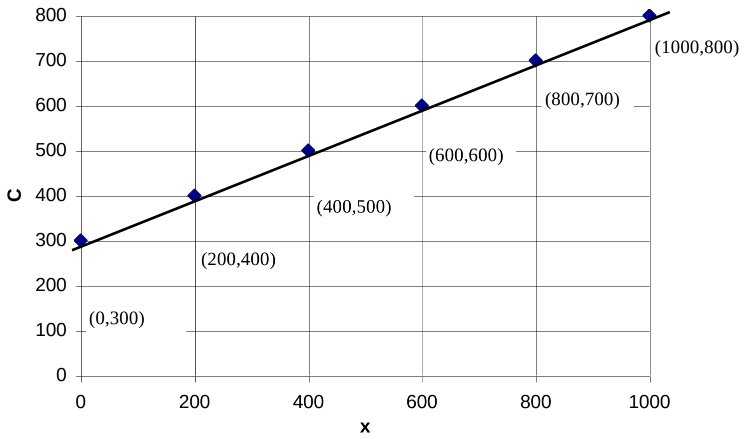
(iii) $y = 4 - 3x$ (iv) $y = 2x - 3$

Solutions

1. (i)

x	0	200	400	600	800	1000
C	300	400	500	600	700	800

(ii)

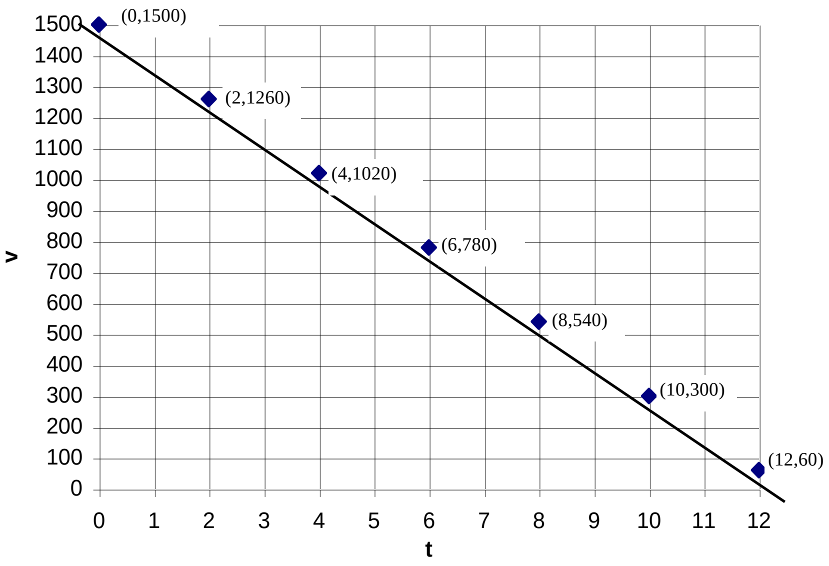


(iii) 900 kg.

2 (i)

t	0	2	4	6	8	10	12
v	1500	1260	1020	780	540	300	60

(ii)

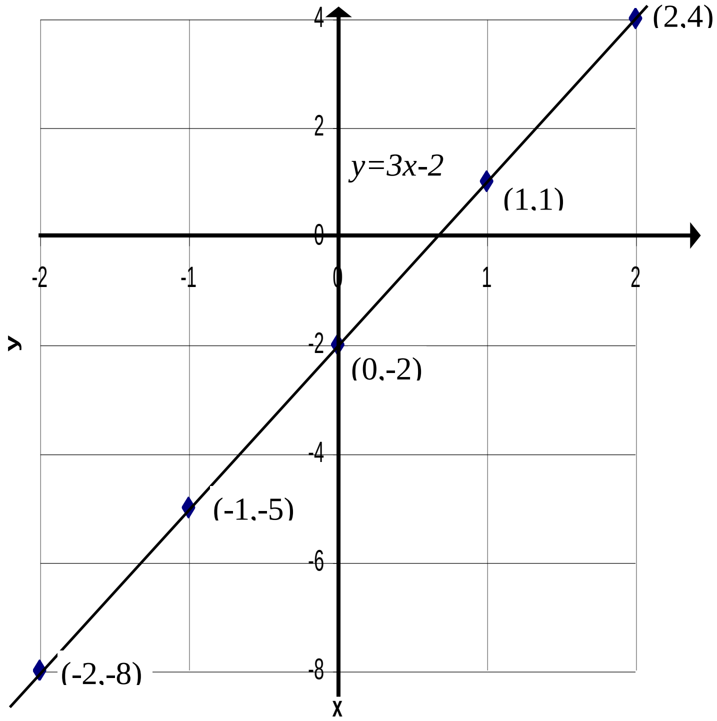


(iii) 12½ years

(iv) 4.2 years

3.

x	-2	-1	0	1	2
y	-8	-5	-2	1	4



4. (i)

x	-3	-2	-1	0	1	2	3
y	-7	-5	-3	-1	1	3	5

(ii)

x	-3	-2	-1	0	1	2	3
y	-1	0	1	2	3	4	5

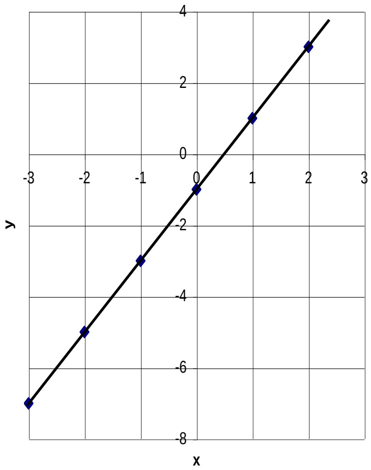
(iii)

x	-3	-2	-1	0	1	2	3
y	13	10	7	4	1	-2	-5

(iv)

x	-3	-2	-1	0	1	2	3
y	-9	-7	-5	-3	-1	1	3

(i)



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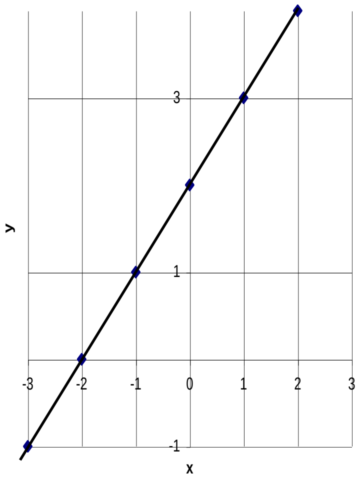
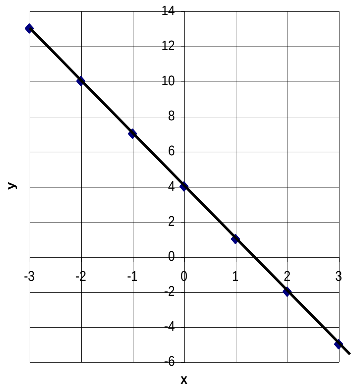
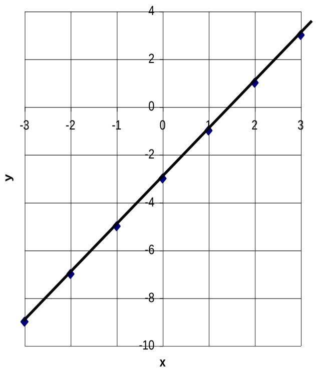


(iii



(iv)



Equations of Lines That Pass Through the Origin

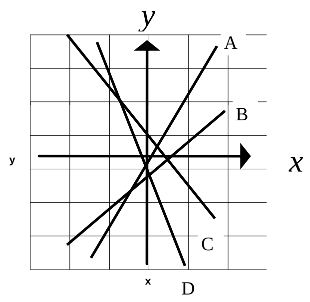
Often you may have a set of data that allows you to draw a graph. This may arise as the result of an experiment or a survey. This section deals with finding the equation of a line or the relationship between two variables. This section is restricted to cases where the straight lines pass through the point. (0, 0).

When a straight line is drawn the important feature that is needed to write an equation is its gradient. The gradient of a line describes how steep the slope of the line is. Consider the lines shown in the graph.

Lines A and B have positive gradients while lines C and D have negative gradients.

The gradient of A is greater than the gradient of B because it is steeper .

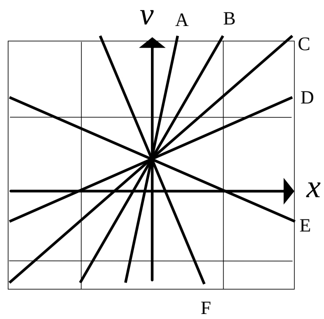
In fact the gradient of A is 2, B is 1, C is -1.5 and D is -3.



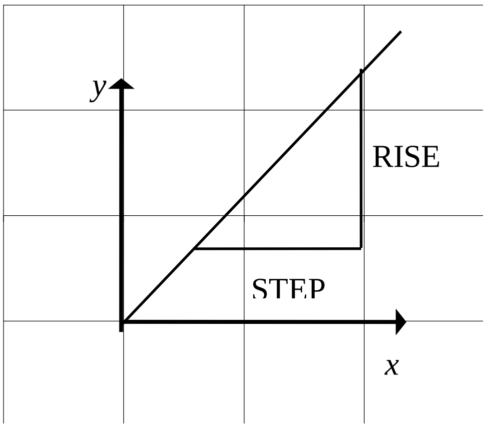
Problem

There are six lines shown in the diagram. Their gradients are listed in the table below, write the correct letter in the spaces on the top row.

Line						
Gradient	5	-½	½	-3	2	1



Calculating Gradients



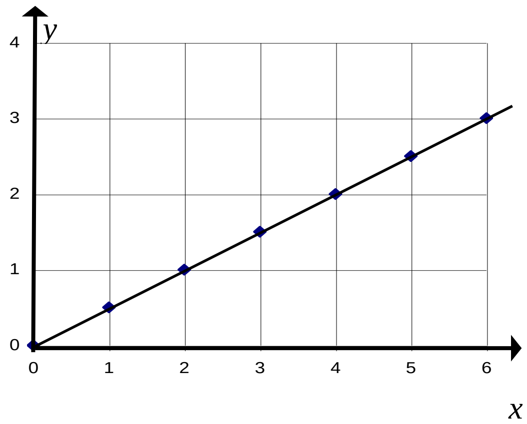
The gradient is calculated by drawing a triangle under the line as shown in the diagram. The two sides of the triangle are often referred to as the rise (vertical side) and the step (horizontal side). The gradient is then given by;

$$\text{gradient} = \frac{\text{Rise}}{\text{Step}}.$$

The letter *m* is often used to refer to the gradient of a line.

If a line passes through (0,0) and has gradient *m*, then its equation is $y = mx$. When two quantities *x* and *y* are related in this way they are said to be proportional.

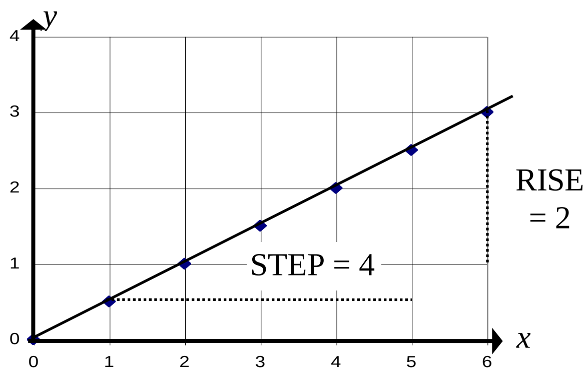
Example 1



To find the gradient of the graph shown it is first necessary to draw a triangle underneath the line. Note that the triangle has been drawn using lines on the graph paper.

The lengths of two sides of the triangle are then found as shown in the second diagram. Now the gradient, *m*, can be calculated using:

$$m = \frac{\text{rise}}{\text{step}} = \frac{2}{4} = \frac{1}{2}$$



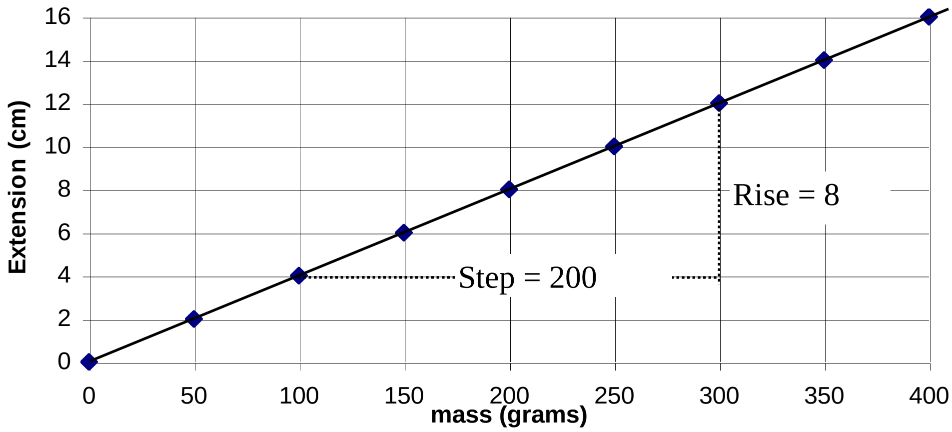
Since the gradient of the line is $\frac{1}{2}$ the equation of the line must be $y = \frac{1}{2}x$.

Example 2

The table below gives the results of an experiment where the extension of a spring was measured when different masses were hung on the spring.

Mass (grams)	0	50	100	150	200	250	300	350	400
Extension (cm)	0	2	4	6	8	10	12	14	16

The data was plotted on the graph below and a triangle drawn so that the gradient can be found. The sides of the triangle have been written in. Note that the lengths must be found using the scales on the side of the graph.



So the gradient, m, can now be found;

$$m = \frac{\text{rise}}{\text{step}} = \frac{8}{200} = \frac{1}{25} \text{ or } 0.04.$$

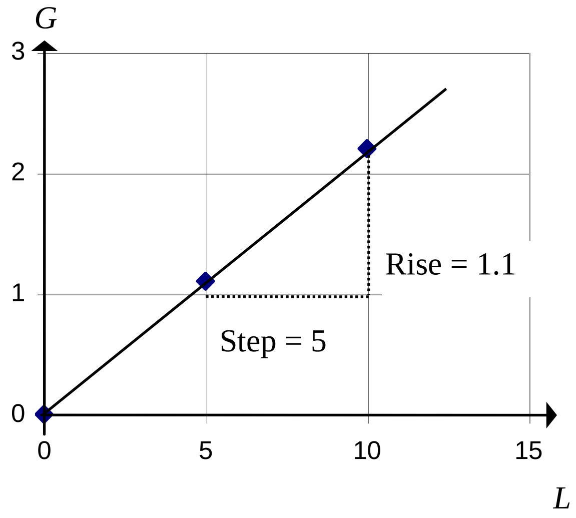
So the equation of the line will be $E = 0.04x$ or $E = \frac{1}{25}x$, where E denotes extension in cm and x mass in grams.

Using the equation the extension for other masses can be calculated. For example if $m = 180$ grams, then;

$$E = \frac{1}{25} \times 180 = 7.2 \text{ cm.}$$

Example 3

The graph shown can be used for converting volumes measured in litres to gallons. On the vertical axis is the volume in gallons G and on the horizontal the volume in litres L.



A triangle has been drawn under the line so that the gradient can be calculated. The gradient, m is given by;

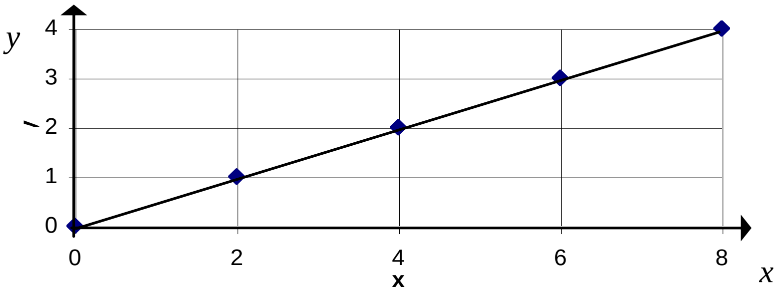
$$\begin{aligned} m &= \frac{\text{rise}}{\text{step}} \\ &= \frac{1.1}{5} \\ &= 0.22 \end{aligned}$$

So the equation of the line is $G = 0.22L$. This can now be used, to convert from litres to gallons. For example to convert 50 litres into gallons, use,

$$\begin{aligned} G &= 0.22\text{L} \\ &= 0.22 \times 50 \\ &= 11 \text{ gallons.} \end{aligned}$$

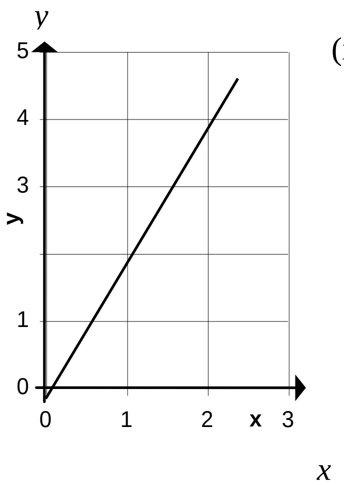
Exercise

1. By drawing a suitable triangle check that the gradient of the line is ½. Write down the equation of the line

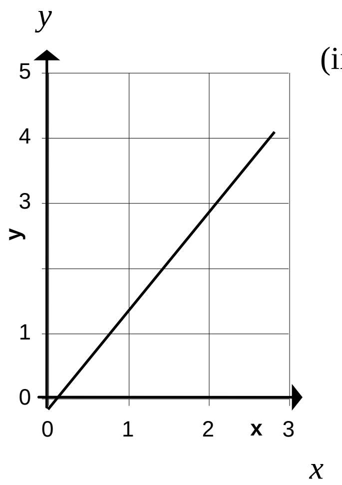


2. For each graph below, find the gradient and the equation of the line.

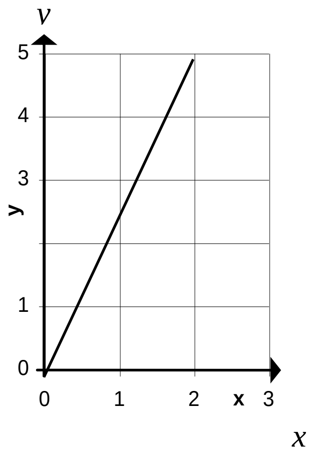
(i)



(ii)



(iii)



3. A straight line passes through the points (0,0) and (4,6).

   (i) Plot these points on a graph and draw a straight line through them.

   (ii) Find the gradient of the line and its equation.

   (iii) Repeat steps (i) and (ii) for the lines that pass through the points specified below.

   (a) (0,0) and (3,9).
   (b) (0,0) and (5,8).
   (c) (0,0) and (6,3).
   (d) (0,0) and (8,1).

4. The table below shows the weekly sales and salary of a salesperson.

Sales (*N*)	50	120	90	150	160

Salary (£) (S)	185	444	333	555	592

(i) Plot a graph of salary (S) (on the vertical) against sales (N) on the (horizontal).

(ii) Find the gradient of the line that you have drawn, and write down the equation of the line.

(iii) How much is paid for each sale?

5. A farm uses the table below when applying insecticide to growing crops.

Area (acres)	5	10	20	25	50
Insecticide (gallons)	21	42	84	105	210

(i) Plot a graph of Insecticide (I) (gallons) against Area (A) (acres).

(ii) Find the gradient of the graph, and write down its equations.

(iii) How much insecticide will be needed for 18 acres?

6. In each case below draw a conversion graph and find its equation.

	Vertical Quantity	Horizontal Quantity	Information
(i)	Pints (P)	Litres (L)	10 pints = 6 Litres
(ii)	Pounds (P)	Kilograms (K)	10 pounds = 22 kg
(iii)	Centimetres (C)	Inches (I)	12 inches = 30 cm
(iv)	Pounds (P)	Dollars (D)	£100 = 180$

Solutions

1. $y = ½x$

2. (i) 2, $y = 2x$ (ii) 1.5, $y = 1.5x$ (iii) 2.5, $y = 2.5x$

3. (ii) 1.5, $y = 1.5x$.

(iii) (a) 3, $y = 3x$.
(b) 1.6, $y = 1.6x$.
(c) 0.5, $y = 0.5x$.
(d) 0.125, $y = 0.125x$.

4. (ii) 3.7, $S = 3.7N$.

(iii) £3.70

5. (ii) 4.2, $I = 4.2A$

(iii) 75.6 gallons

6. (i) $P = 1.67L$

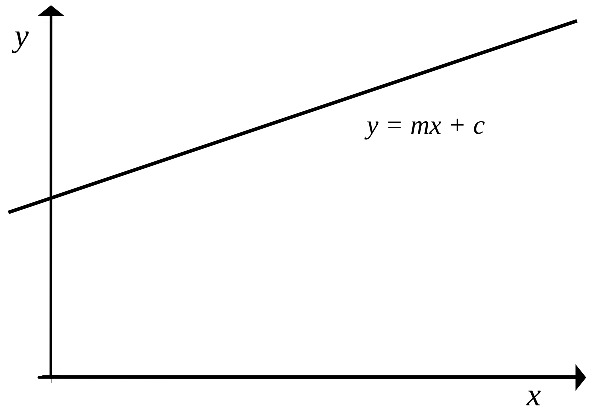
(ii) $P = 0.45K$

(iii) $C = 2.5I$

(iv) $P = 0.56D$

The Equation of Any Straight Line

Very often straight lines will not pass through (0,0) as in the example illustrated. To write the equation of such a line it is important to take account of both the gradient of the line and the point where it crosses the vertical axis, known as the intercept. If the gradient is m and the intercept is c, then the equation of the line is;



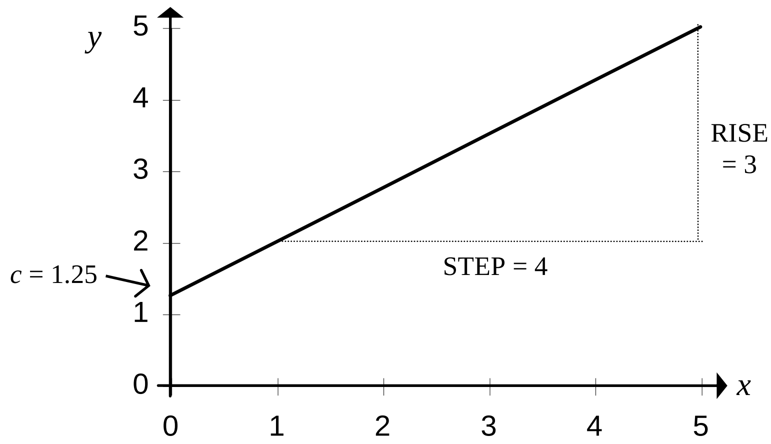
$$y = mx + c.$$

When the graph of two quantities forms a straight line, they are said to have a linear relationship.

Example 1

To find the equation of the graph shown, it is first necessary to find the gradient of the line. Drawing a triangle under the line allows this to be calculated. The gradient, m, is given by;

$$m = \frac{\text{Rise}}{\text{Step}} = \frac{3}{4} = 0.75.$$

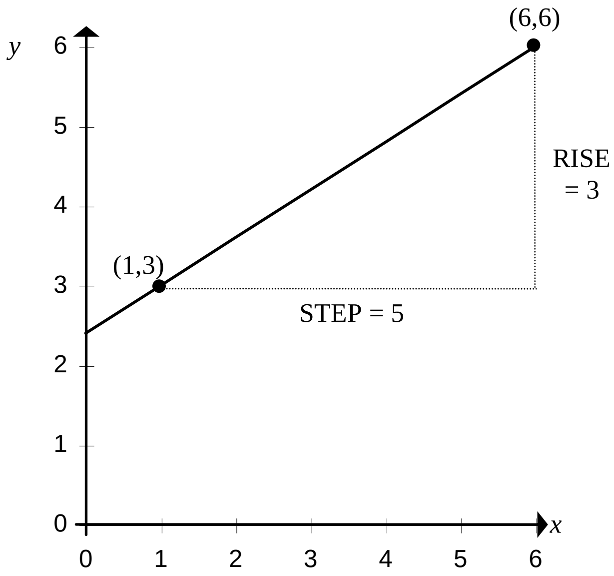


The intercept, c, is the y - value where the line crosses the vertical axis. In this case that value is 1.25, so $c = 1.25$. Now the equation can be written down,

$$y = mx + c = 0.75x + 1.25.$$

Example 2

A line is known to pass through the two points (1,3) and (6,6). To find its equation first plot these two points and draw a straight line through them.



To find the equation of this line draw a triangle under the line to find the gradient and note where it crosses the vertical axis. The gradient, *m*, is then,

$$m = \frac{\text{Rise}}{\text{Step}} = \frac{3}{5} = 0.6$$

The intercept, *c*, is 2.4. Now the equation is;

$$y = mx + c$$

so,

$$y = 0.6x + 2.4$$

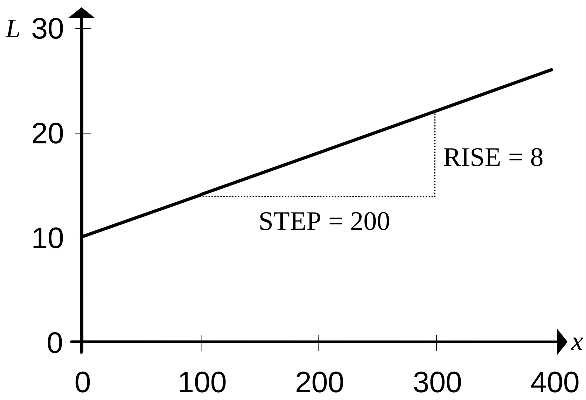
Example 3

When a mass is hung on a spring it stretches. The data in the table below was collected during an experiment.

Mass (grams)	0	100	200	300	400
Length cm	10	14	18	22	26

To find a relationship between the length (*L*) and the mass (*x*) first plot of a graph of *L* against *m*. As this produces a straight line the equation of the line will be of the form;

$$L = mx + c$$



where m is the gradient and *c* the intercept. The gradient, *m*,can be calculated using the triangle shown,

$$m = \frac{\text{Rise}}{\text{Step}} = \frac{8}{200} = 0.04.$$

The intercept, *c*, is 10, so the equation of the line is,

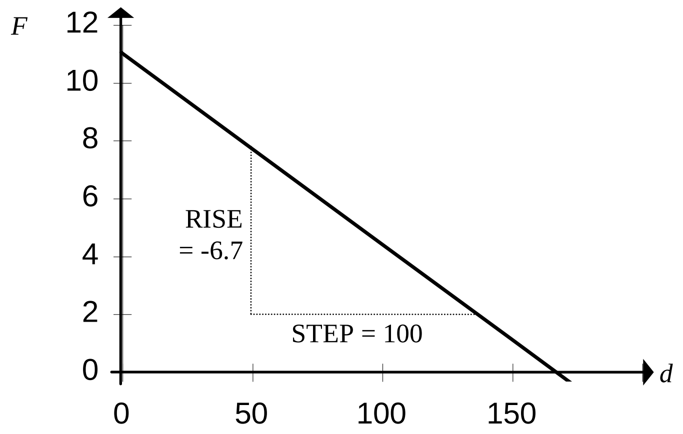
$$L = mx + c$$
$$= 0.04x + 10$$

Example 4

A drive monitors how much fuel has been used on a long journey. The observations made are recorded in the table below.

Distance Travelled (Miles)	0	50	100	150
Fuel in Tank (Gallons)	11.0	7.7	4.3	1.0

To find the relationship between these two quantities first plot the points and draw a straight line through them as shown in the diagram.



Note that as the line slopes down as it moves to the left the gradient will be negative. The gradient, m, can be calculated as;

$$m = \frac{\text{Rise}}{\text{Step}}$$

$$= \frac{-6.7}{100}$$

$$= -0.067.$$

The intercept is 11 and so the equation of the line becomes,

$$F = 0.067d + 11$$

or

$$F = 11 - 0.067d,$$

When F is the fuel used and d the distance travelled.

Using the equation to calculate the amount of fuel left after 150 miles give;

$$F = 11 - 0.067 \times 150$$

$$= 0.95.$$

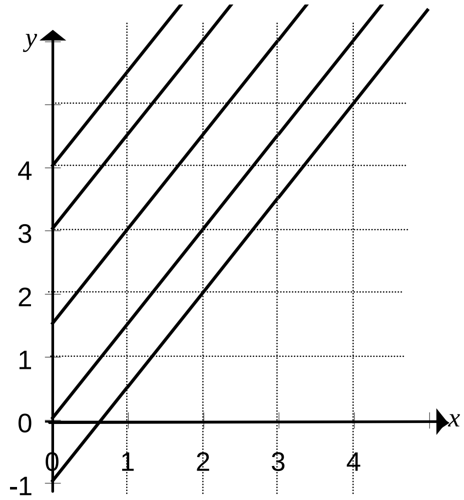
This is not quite equal to the value 1 given in the table. This is due to the fact that the numbers given in the table have been rounded to 1 decimal place.

Exercise

1. (i) Find the gradient of each of the lines shown.

   (ii) For each line write down the intercept.

   (iii) Write down the equation of each line.

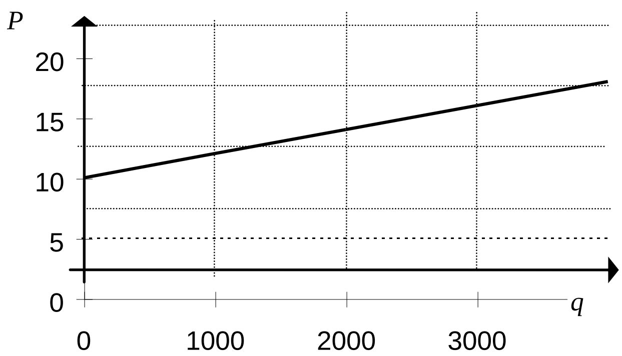


2. The graph shows the results of an experiment, with a straight line drawn through the points that have been plotted.

   (i) Find the gradient of the line.

   (ii) Find the intercept.

   (iii) Write down the equation of the line in the form $P = mq + c$.



3. Plot the two points given in each example below, draw a line through the points, find its gradient and intercept it then write down the equation of the line.

(i)	(0,2) (5,5)	(ii)	(3,2) (5,6)
(iii)	(1,7) (4,9)	(iv)	(1,1) (4,3)
(v)	(0,5) (4,2)	(vi)	(1,6) (4,2)

4. The tables below show how two quantities are related. In each case plot the points, draw a straight line through them and find the equation of the line, in the form given.

   (i)

x	0	100	200	300	400
L	6	6.4	6.8	7.2	7.6

$L = mx + c$

   (ii)

n	0	10	20	30	40	50
s	75	175	275	375	475	575

$S = mn + c$

5. The cost of manufacturing 10 typewriters per day is £350, while it costs £650 to produce 20. Find the linear relationship between the cost, C, and the number, n, produced in the form $C = mn + c$.

6. The cost of manufacturing 100 cameras per week is £700, and for 120 cameras per week £800. Find the linear relationship between the cost, C, and the number, n, of cameras produced.

7. A catering company charge £250 for a reception for 10 people and £400 for 25. Find a relationship between the price, P, and the number of people, x, in the form $P = mx + c$. What would a reception for 40 people cost?

8. In the calibration of a thermometer, the distance, L, from the bulb to the top of the mercury column is measured at various temperatures, T. Plot a graph of L on the vertical against T on the horizontal for the data below.

T °C	5	20	35	50	80
L (mm)	5.00	20.06	35.22	50.18	80.30

Verify that there is a linear relationship between L and T. Express this relationship in the form $L = mT + c$.

Solutions

1. (i) 1.5

(ii) A, 4; B, 3; C, 1.5; D, 0; E, -1.

(ii) A, $y = 1.5x + 4$; B, $y = 1.5x + 3$;
C, $y = 1.5x + 1.5$; D $y = 1.5x$;
E, $y = 1.5x - 1$.

2. (i) 0.002

(ii) 10

(iii) $P = 0.002q + 10$

3. (i) $y = 0.6x + 2$

(ii) $y = 2x - 4$

(iii) $y = \frac{2}{3}x + 6\frac{1}{3}$

(iv) $y = \frac{2}{3}x + \frac{1}{3}$

(v) $y = 5 - 0.75x$

(vi) $y = 7\frac{1}{3} - 1\frac{1}{3}x$

4. (i) $L = 0.004x + 6$

(ii) $S = 10n + 75$

5. $C = 30n + 50$

6. $C = 5n + 200$

7. $P = 10x + 150$, £550.

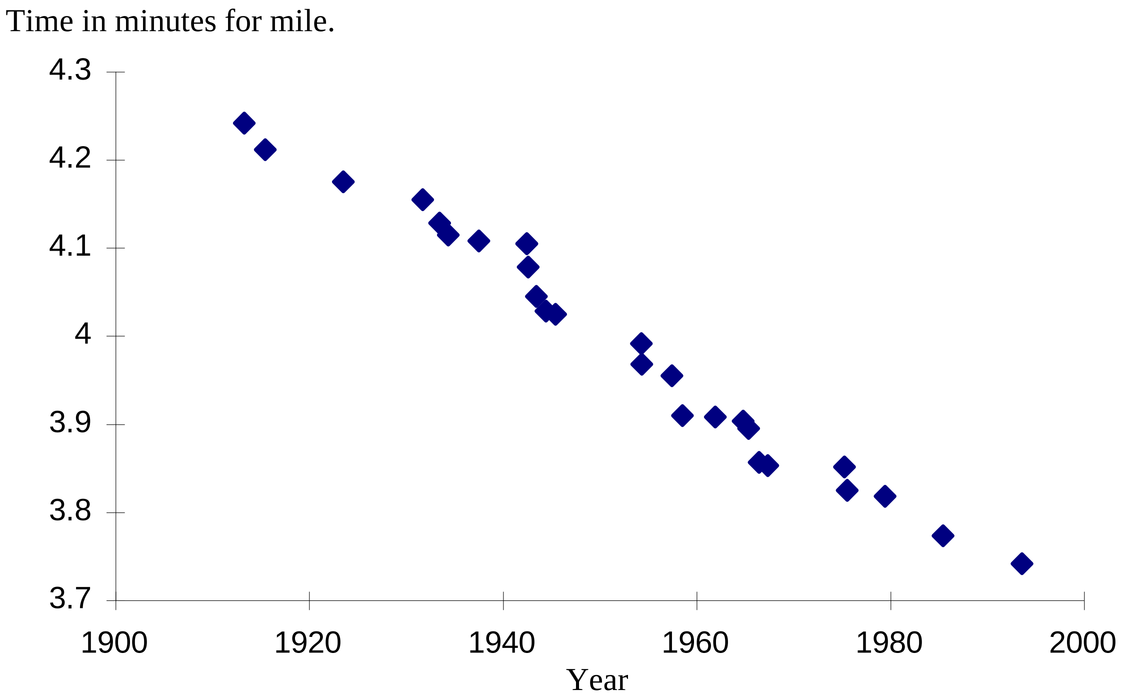
8. $L == 1.004T - 0.02$

Lines of Best Fit

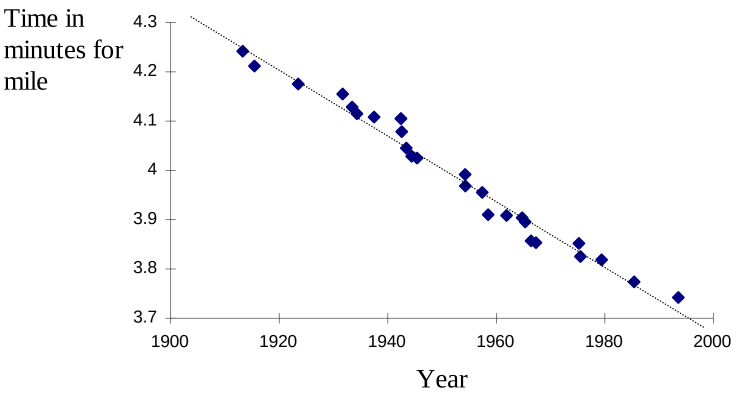
Graphs of experimental data or data that has been collected from observations of real events will rarely lie exactly on a straight line, but can often be described well by the use of a straight line. The example below illustrates how a line of best fit can be drawn and also warns of the dangers of extending a line too far!

Example 1

The world record for the mile has gradually decreased with time. One of the most significant records was the first that broke the four minute barrier set by Roger Bannister in 1954. The graph shows new records set between 1913 and 1980.



Note how the trend is downwards and that the points follow a trend that could be modelled by a straight line. The graph below shows a line fitted to the same data. This line is known as a line of best fit.



Lines of best fit can be calculated and many computer packages have the facilities to do this, but they can also be approximately drawn " by eye", with the aim of minimising the distances of all the points from the line.

Use the graph above to estimate when the world record will drop to 3.7 minutes.

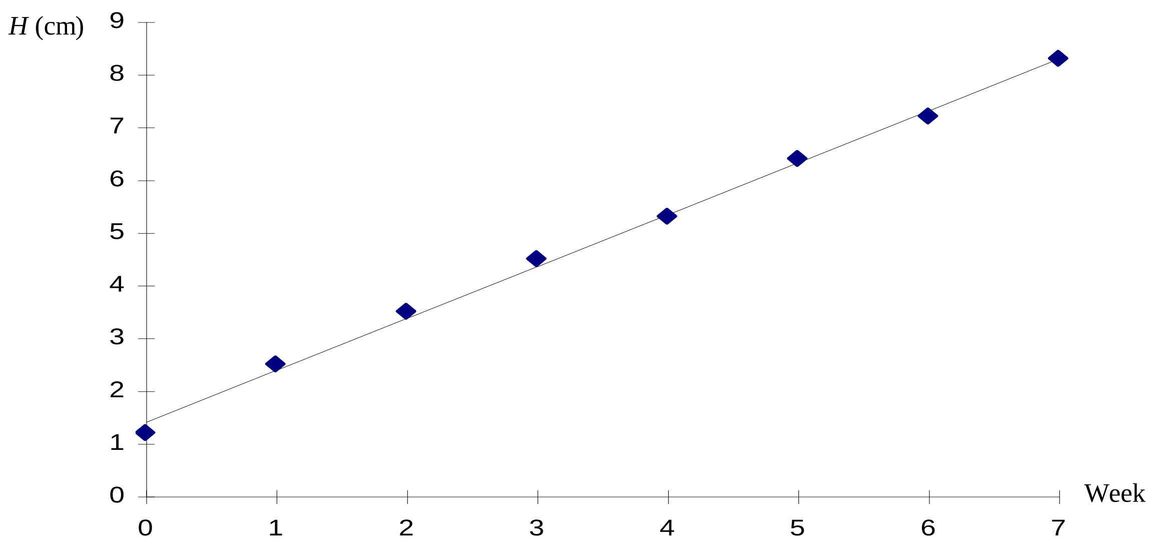
If the graph were extended further it would be possible to predict that in about 2500 you would be able to run a mile so quickly that you could finish before your started!!! It is important to be aware of the dangers of extending a line too far.

Example 2

A biologist measured the height of a small plant at weekly intervals and obtained the results recorded in the table.

Week	0	1	2	3	4	5	6	7
Height (cm)	1.2	2.5	3.5	4.5	5.3	6.4	7.2	8.3

The graph below shows the data with a line of best fit.



The line shown has a gradient that is approximately 1 and an intercept of 1.4. So the equation of the line is

$$H = n + 1.4$$

where H represents height and n the number of weeks growth. This equation can then be used to make predictions.

Exercise

1. In an experiment a flask of water is heated. The temperature of the water is recorded at two minute interval. The results are recorded in the table below.

Time (minutes)	0	2	4	6	8	10
Temperature C°	18°	30°	42°	56°	71°	84°

(a) Plot the data on a graph and find the equation of the line of best fit.
(b) Use the equation to predict the temperature after 11 minutes.

2. A driver records the petrol consumed on a number of journeys of different lengths. The data is presented in a table below.

Journey Length (miles)	100	180	250	300	320	350
Petrol Consumption (gallons)	3.5	5.6	7.9	8.4	9.3	10.9

Plot a graph of petrol consumed (vertical axis) against journey length (horizontal axis) and find the equation of the line of best fit. Use this to predict the petrol needed for a journey of 280 miles.

3. The information presented in the last part of the graph used in the introduction is presented in the table below.

Year	1984	1987	1988	1989	1991	1992	1994
No of Triplet & higher order births per 100,00	13	21	20	29	32	31	40

Plot a graph to illustrate this data and draw a line of best fit. By extending your line predict the number of Triplet or Higher order births that will take place in the year 2000.

4. A long distance driver records the times that it takes to make journeys of different lengths. This information is recorded below.

Journey Length (miles)	150	220	260	290	320
Time Taken (Hours)	3¼	4½	6¼	6½	7¾

(a) Comment on the way that the driver records the time taken.
(b) Plot the data and draw a line of best fit.
(c) Find the equation of the line of best fit.

Solutions

Note that you will not get exact answers to these questions.

1. (a) Gradient = 6.7.
   Intercept = 17.
   $y = 6.7x + 17$.

   (b) $90^\circ C$

2. Gradient = 0.03.
   Intercept = 0.7.
   $y = 0.03x + 0.7$.
   9 gallons

3. 55.

4. (a) To nearest ¼ hour.
   (b) Gradient = 0.025.
   Intercept = –1.
   $y = 0.025x - 1$

Mathematics Support Series

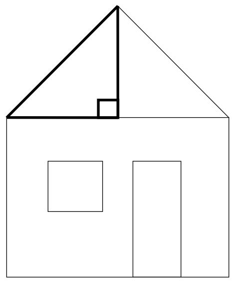
Basic Trigonometry

Ted Graham
Centre for Teaching Mathematics
University of Plymouth

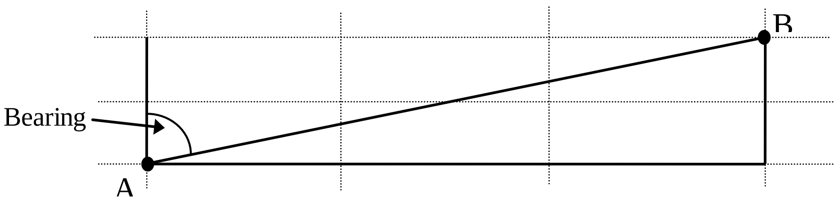
(maths236/dlh)

Introduction

Trigonometry can be used in any right angled triangle to find lengths or angles. This can be important in design tasks, such as finding the quantity of tiles or slates to roof a house. The diagram shows a triangle that could be used for this purpose.



Two grid positions may be known as a map. The right angled triangle shown in the diagram could be used to find bearings or lengths.



This unit is concerned with simple trigonometry. This can be used in right angled triangles to calculate angles or lengths. This unit will help you to:

(i) find lengths using trigonometry,

(ii) find angles using trigonometry,

(iii) select sin, cos or tan to solve a given problem.

Using Sine in Right Angled Triangles

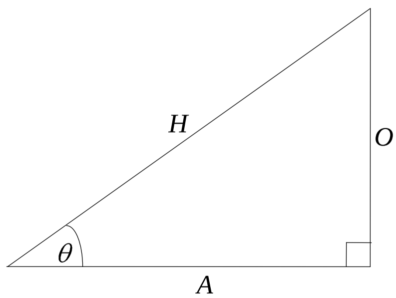
When using any trigonometric functions in a right angled triangle it is important to use a consistent notation. The sides of the triangles are labelled with reference to the angle that is to be used.

The longest side of the triangle is always known as the hypotenuse or *H*

The Greek letter θ is often used to represent the angle.

The side next to the angle θ is known as the adjacent *A* .

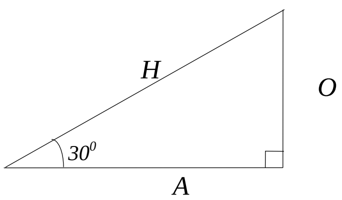
The side opposite the angle θ is known as the opposite *O*.

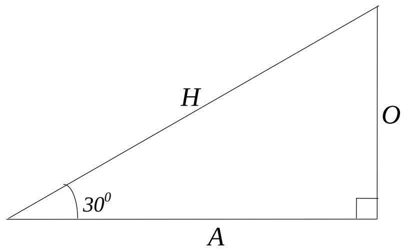


The sine function relates the opposite side to the hypotenuse for any right angled triangle. The sine of an angle is given by:

$$\sin\theta = \frac{O}{H}$$

For any fixed angle this means that the lengths of the opposite side at the hypotenuse are in the same ratio. The two triangles below have an angle of 30^o marked. For each one measure the opposite and the hypotenuse.

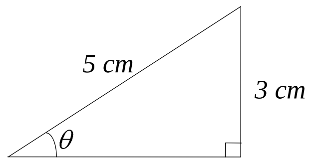




For each triangle verify that sinθ = 0.5. This means that the opposite side is half the length of the hypotenuse if the angle is 30^o. For a different angle there will be a different ratio.

Finding Angle with Sine

Example 1



The triangle shown has two sides that are known and the angle θ that is unknown. Using sine the angle θ can be calculated.

Here the hypotenuse has length 5 cm, so H = 5 and the opposite side has length 3 cm so O = 3.

Using:

$$\sin\theta = \frac{O}{H}$$

gives

$$\sin\theta = \frac{3}{5}$$

$$= 0.6.$$

The sine of the angle is equal to 0.6. To find the angle that corresponds to this you need a scientific calculator. First enter 0.6 and then press the $\sin^{-1}$ key. You may have to use a 2nd F or INV key at the same time. Your calculator will respond with 36.9°. The key sequence is shown below:

[0] [.] [6] [Inv] or [2ndF] [$\sin^{-1}$]

Example 2

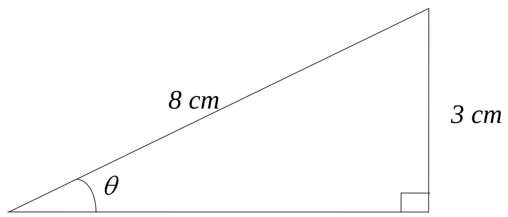
Find the angle θ in the triangle shown.

Here H = 8 and O = 3,

so

$$\sin\theta = \frac{3}{8}$$

$$= 0.375.$$



The angle θ is then given by;

$$\theta = \sin^{-1}(0.375)$$

$$= 22.0^{\circ}$$

Remember to use the $\sin^{-1}$ key on your calculator.

Problems

1. Use your calculator to find the angle θ in each case below.

(a) $\sin\theta = 0.5$

(b) $\sin\theta = 0.82$

(c) $\sin\theta = 0.414$

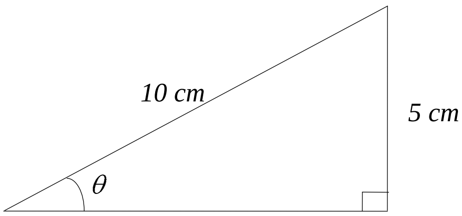
(d) $\sin\theta = 0.62$

(e) $\sin\theta = 0.125$

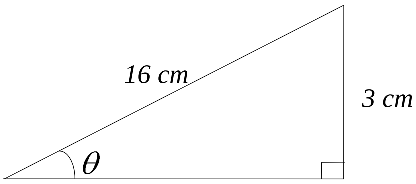
(f) $\sin\theta = 0.012$.

2. Find the angle marked θ in each triangle below.

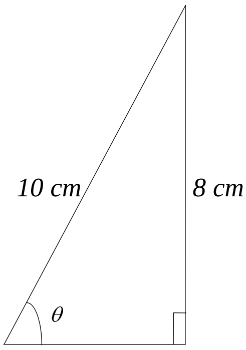
(a)



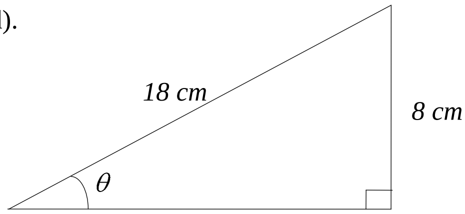
(b)



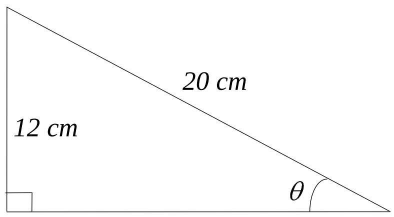
(c)



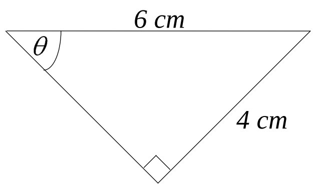
(d).



(e)



(f)



Solutions

1. (a) $\theta = 30^o$
   (b) $\theta = 55.1^o$
   (c) $\theta = 24.5^o$
   (d) $\theta = 38.3^o$
   (e) $\theta = 7.2^o$
   (f) $\theta = 0.7^o$

   Answers are all given to 1 decimal place.

2. (a) $\sin\theta = \frac{5}{10}$

   $= 0.5$

   $\theta = 30^o$

   (b) $\sin\theta = \frac{3}{16}$

   $= 0.1875$

   $\theta = 10.8^o$

   (c) $\sin\theta = \frac{8}{10}$

   $= 0.8$

   $\theta = 53.1^o$

   (d) $\sin\theta = \frac{8}{18}$

   $= 0.444$

   $\theta = 26.4^o$

   (e) $\sin\theta = \frac{12}{20}$

   $= 0.6$

   $\theta = 36.9^o$

   (f) $\sin\theta = \frac{4}{6}$

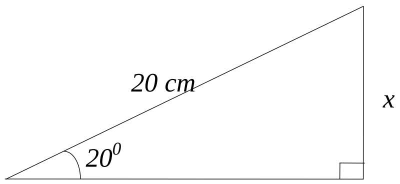
   $= 0.667$

   $\theta = 41.8^o$

Finding the Opposite Side with Sine

Example 1

If an angle is known it is possible to find the length of the opposite side using sine. Consider the triangle below, where x is the length that is to be calculated.



To find the opposite side, x, first note $\theta = 20^o$ and H = 20 cm, then use:

$$\sin\theta = \frac{O}{H}$$

to give $$\sin 20^o = \frac{x}{20}.$$

This is a simple equation and to find x it is simply necessary to multiply both sides by 20 to give,

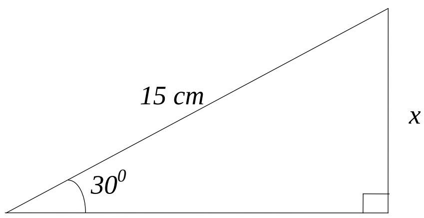
$$20 \times \sin 20^o = x.$$

On most calculators this can be found using the sequence of keys below.

2	0	×	2	0	sin	=

Check that you can obtain x = 6.84 cm.

Example 2



Find the length marked x in the triangle.

Note that $\theta = 30^o$, H = 15 cm and $O = x$.

Using $$\sin\theta = \frac{O}{H}$$

gives $$\sin 30^o = \frac{x}{15}$$

Multiplying both sides of this equation by 15 gives

$$15 \times \sin 30^o = x$$

$$x = 7.5 \text{ cm}.$$

Check that you can obtain 7.5 on your own calculator.

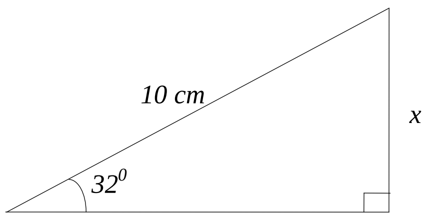
Problems

1. Use your calculator to find sinθ for each case given below.

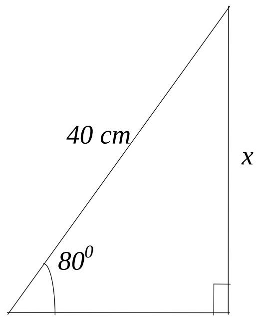
(a)	sin 45^o	(d)	sin 80^o
(b)	sin 37^o	(e)	sin 40^o
(c)	sin 64^o	(f)	sin 25^o

2. For each triangle below find the length of the side marked x.

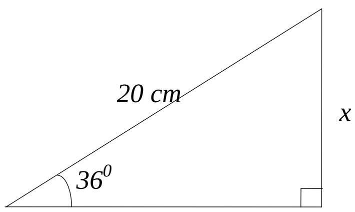
(a)



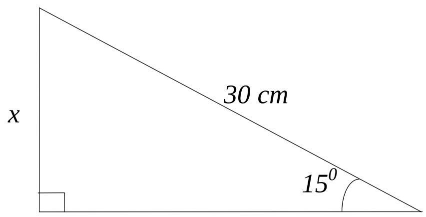
(d)



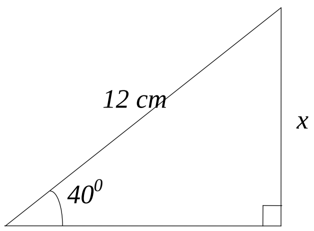
(b)



(e)



(c)



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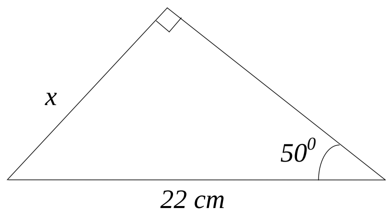


Solutions

1. (a) 0.707 (d) 0.985
   (b) 0.602 (e) 0.643
   (c) 0.899 (f) 0.423

   All answers to 3 decimal places.

2. (a) $\sin 32^o = \frac{x}{10}$

   $x = 10 \times \sin 32^o$

   $= 5.3$ cm

   (d) $\sin 80^o = \frac{x}{40}$

   $x = 40 \times \sin 80^o$

   $= 39.4$ cm

   (b) $\sin 36^o = \frac{x}{20}$

   $x = 20 \times \sin 36^o$

   $= 11.8$ cm

   (e) $\sin 15^o = \frac{x}{30}$

   $x = 30 \times \sin 15^o$

   $= 7.8$ cm

   (c) $\sin 40^o = \frac{x}{12}$

   $x = 12 \times \sin 40^o$

   $= 7.7$ cm

   (f) $\sin 50^o = \frac{x}{22}$

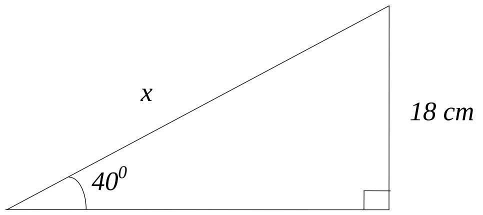
   $x = 22 \times \sin 50^o$

   $= 16.9$ cm.

Finding The Hypotenuse With Sine

Example 1

It is possible to find the length of the hypotenuse if an angle and the opposite side are both known. In the triangle below the hypotenuse is the side to be calculated and has been marked x.



So $\theta = 40^o$, O = 18 cm and $H = x$.

Using

$$\sin\theta = \frac{O}{H}$$

gives $$\sin 40^o = \frac{18}{x} \qquad (1)$$

To solve this equation first multiply both sides of the equation by x to give,

$$x \times \sin 40^o = 18.$$

Then divide both sides of the equation by $\sin 40^o$ to give,

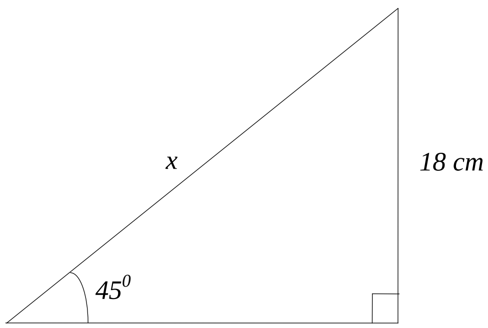
$$x = \frac{18}{\sin 40^o} \qquad (2)$$

$$= 28.0 \text{ cm}.$$

Check that you can obtain this result on your own calculator. Note how in rearranging the equation labelled (1) the position of x and $\sin 40^o$ are swapped to give equation (2).

Example 2

Find the length marked x in the triangle in the diagram.



First note

$\theta = 45^o$
O = 18 cm
$H = x$

Using $$\sin\theta = \frac{O}{H}$$

gives $$\sin 45^o = \frac{18}{x}.$$

Rearranging this equation then gives,

$$x \times \sin 45^o = 18$$

and

$$x = \frac{18}{\sin 45^{\circ}}$$

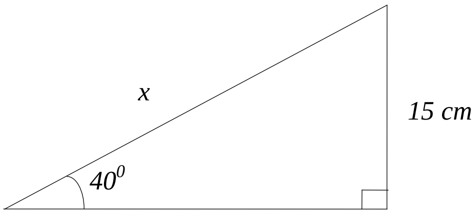
$$= 25.5 \text{ cm.}$$

Check that you can obtain 25.5 on your own calculator.

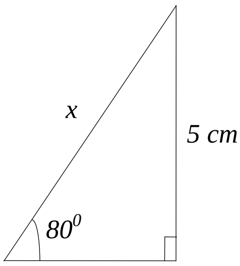
Problems

1. For each triangle below find the length of hypotenuse which has been labelled x in each case.

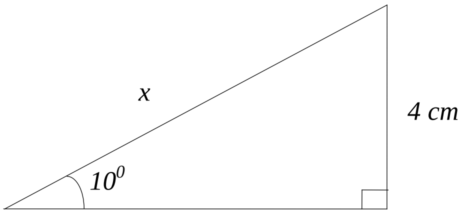
(a)



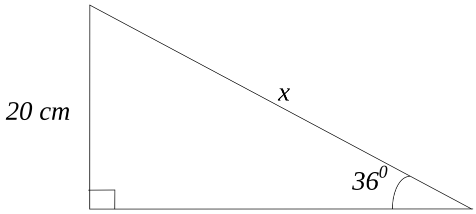
(b)



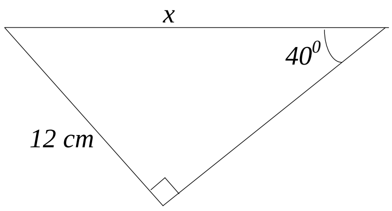
(c)



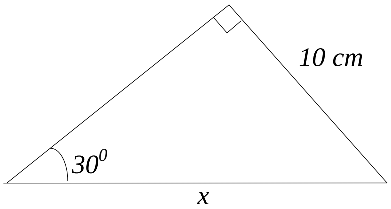
(d)



(e)

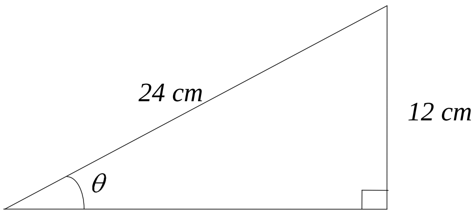


(f)

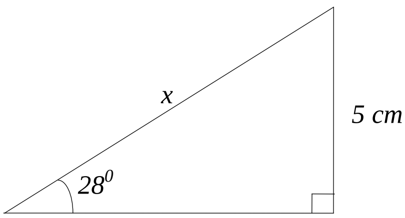


2. For each triangle below find either the length marked x or the angle marked θ.

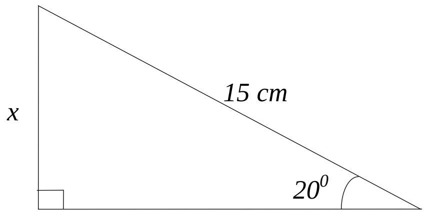
(a)



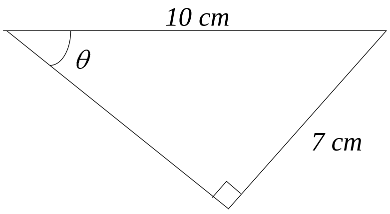
(b)



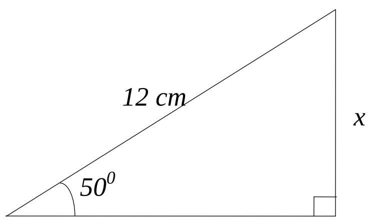
(d)



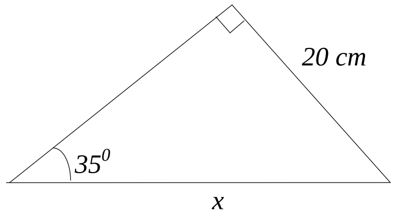
(e)



(c)



(f)



Solutions

1. (a) $\sin 40^o = \dfrac{15}{x}$

$x = \dfrac{15}{\sin 40^o}$

=23.3 cm

(b) $\sin 80^o = \dfrac{5}{x}$

$x = \dfrac{5}{\sin 80^o}$

= 5.1 cm

(c) $\sin 10^o = \dfrac{4}{x}$

$x = \dfrac{4}{\sin 10^o}$

= 23.0 cm

(d) $\sin 36^o = \dfrac{20}{x}$

$x = \dfrac{20}{\sin 36^o}$

= 34.0 cm.

(e) $\sin 40^o = \dfrac{12}{x}$

$x = \dfrac{12}{\sin 40^o}$

= 18.7 cm.

(f) $\sin 30^o = \dfrac{10}{x}$

$x = \dfrac{10}{\sin 30^o}$

= 20.0 cm.

2. (a) $\sin \theta = \dfrac{12}{24}$

= 0.5

$\theta = 30^o$

(Use $\sin^{-1}$ on calculator)

(d) $\sin 20^o = \dfrac{x}{15}$

$x = 15 \times \sin 20^o$

= 5.1 cm.

(b) $\sin 28^\circ = \dfrac{5}{x}$

$x = \dfrac{5}{\sin 28^\circ}$

$= 10.7$ cm

(e) $\sin \theta = \dfrac{7}{10}$

$= 0.7$

$\theta = 44.4^\circ$

(Use $\sin^{-1}$ on calculator)

(c) $\sin 50^\circ = \dfrac{x}{12}$

$x = 12 \times \sin 50^\circ$

$= 9.2$ cm

(f) $\sin 35^\circ = \dfrac{20}{x}$

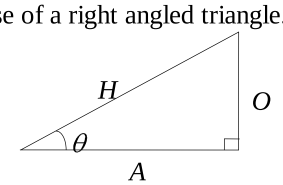
$x = \dfrac{20}{\sin 35^\circ}$

$= 34.9$ cm.

Using Cosine in Right Angled Triangles

The cosine function relates the adjacent and hypotenuse of a right angled triangle. The cosine of an angle is given by;

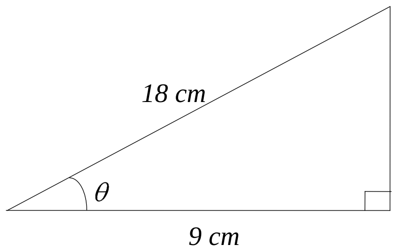
$$\cos\theta = \frac{A}{H} \ .$$



Cosine can be used in a very similar way to sine, but working with the adjacent and the hypotenuse instead of the opposite and the hypotenuse. You need to carefully select when starting a problem whether it is appropriate to use sine or cosine. The examples below show how cosine can be used to find angles and lengths in a triangle.

Example 1 (Finding an angle)

Find the angle marked θ in the triangle.



First note

$A = 9$
$H = 18$

Using $\cos\theta = \frac{A}{H}$

gives $\cos\theta = \frac{9}{18}$

$= 0.5.$

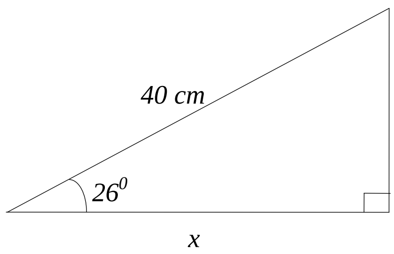
The angle θ can now be found using the $\cos^{-1}$ key on your calculator. In this case this gives:

$\theta = 60^{o}.$

Check that you can obtain this result on your calculator.

Example 2 (Finding the adjacent side)

Find the length marked x in the triangle



First note,

$\theta = 26^{o}$
$A = x$
$H = 40.$

Using $\cos\theta = \frac{A}{H}$

gives $\cos 26^{o} = \frac{x}{40}.$

To find the value of x multiply both sides of this equation to give;

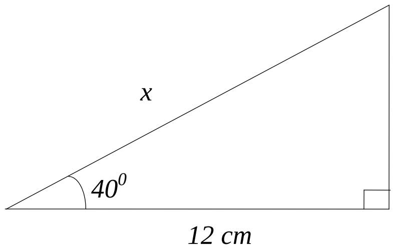
$x = 40 \times \cos 26^{o}$

$= 35.95$ cm.

Check that you can obtain this result on your calculator.

Example 3 (Finding the Hypotenuse)

Find the length marked x in the triangle.



Note that

$\theta = 40^o$
$A = 12$
$H = x$.

Using $\cos\theta = \dfrac{A}{H}$

gives $\cos 40^o = \dfrac{12}{x}$.

Multiplying both sides of the equation by x gives,

$$x \cos 40^o = 12,$$

and then dividing both sides by $\cos 40^o$ gives,

$$x = \frac{12}{\cos 40^o}$$

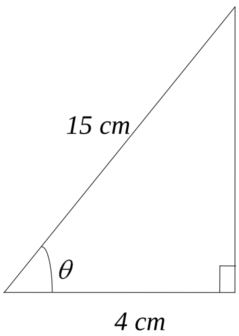
$$= 15.7 \text{ cm}.$$

Check that you can obtain this result.

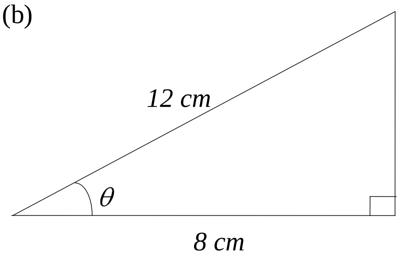
Exercise

1. Find the angle θ in each triangle below.

(a)

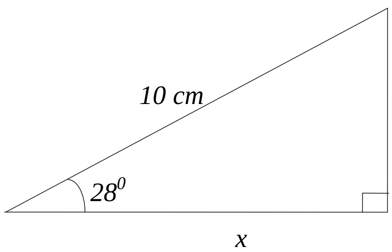


(b)

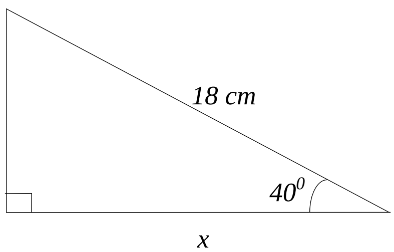


2. Find the length x in each triangle below.

(a)

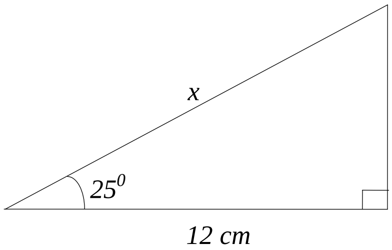


(b)

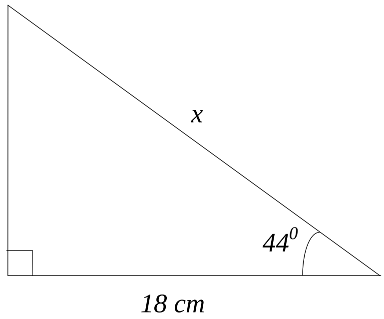


3. Find the length x in each triangle below.

(a)

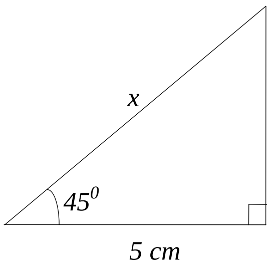


(b)

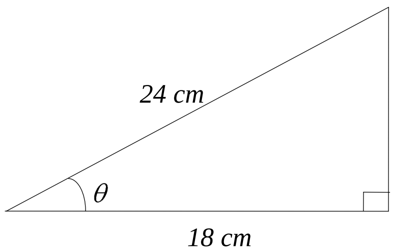


4. Find the angle θ or the length marked x in each triangle below.

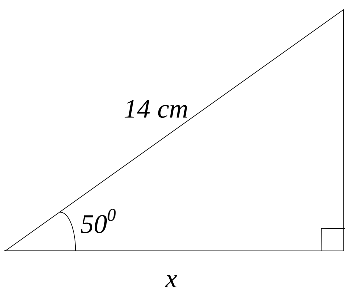
(a)



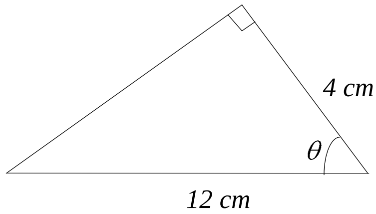
(b)



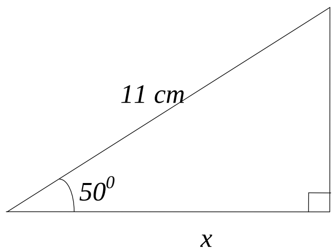
(c)



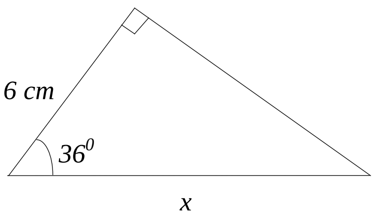
(d)



(e)



(f)



Solutions

1. (a) $\cos\theta = \frac{4}{15}$

$= 0.2667$

$\theta = 74.5^o$

Use $\cos^{-1}$ on calculator

(b) $\cos\theta = \frac{8}{12}$

$= 0.6667$

$\theta = 48.2^o$

(Use $\cos^{-1}$ on calculator.)

2. (a) $\cos 28^o = \frac{x}{10}$

$x = 10 \times \cos 28^o$

$= 8.8$ cm

(b) $\cos 40^o = \frac{x}{18}$

$x = 18 \times \cos 40^o$

$= 13.8$ cm.

3. (a) $\cos 25^o = \frac{12}{x}$

$x = \frac{12}{\cos 25^o}$

$= 13.2$ cm

(b) $\cos 44^o = \frac{18}{x}$

$x = \frac{18}{\cos 44^o}$

$= 25.0$ cm.

4. (a) $\cos 45^o = \frac{5}{x}$

$x = \frac{5}{\cos 45^o}$

$= 7.1$ cm

(b) $\cos\theta = \frac{18}{24}$

$= 0.75$

$\theta = 41.4^o$

(Use $\cos^{-1}$ on calculator).

(c) $\cos 50^o = \frac{x}{14}$

$x = 14 \times \cos 50^o$

$= 9.0$ cm

(d) $\cos\theta = \frac{4}{12}$

$= 0.3333$

$\theta = 70.5^o$

(Use $\cos^{-1}$ on calculator).

(e) $\cos 50^o = \frac{x}{11}$

$x = 11 \times \cos 50^o$

$= 7.1$ cm

(f) $\cos 36^o = \frac{6}{x}$

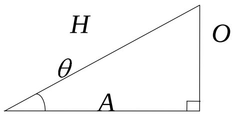
$x = \frac{6}{\cos 36^o}$

$= 7.4$ cm.

Using Tangent in Right Angled Triangles

The tangent relates the opposite and adjacent sides of a right angled triangle. The tangent of an angle is given by,

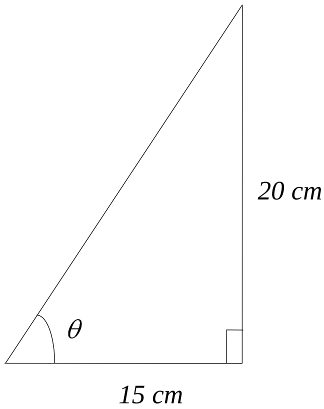
$$\tan\theta = \frac{O}{A}.$$



Tangent can be used in a similar way to sine and cosine but for problems that involve the opposite and adjacent sides. The following examples illustrate how tangent can be used to find lengths or angles.

Example 1 (Finding an angle)

Find the angle θ in the triangle shown.



First note that

$O = 20$
$A = 15$.

Using $\tan\theta = \frac{O}{A}$

gives $\tan\theta = \frac{20}{15}$

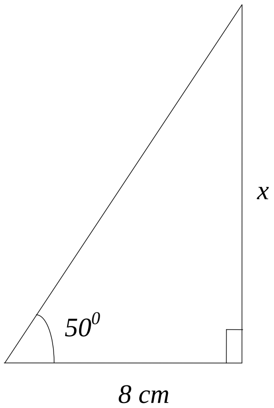
$= 1.333.$

The angle θ can now be found using the $\tan^{-1}$ key on your calculator. This gives $\theta = 53.1^o$.

Check that you can obtain this result on your calculator.

Example 2 (Finding the Opposite Side)

Find the length of the opposite side marked x in the diagram.



First note that

$\theta = 50^o$
$O = x$
$A = 8$.

Then using $\tan\theta = \frac{O}{A}$

gives $\tan 50^o = \frac{x}{8}.$

Then multiplying by 8 gives,

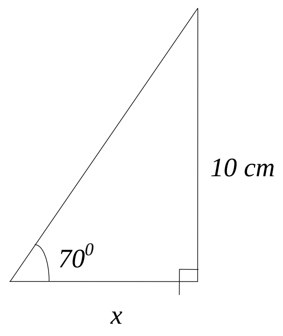
$x = 8 \times \tan 50^o$

$= 9.5$ cm.

Check that you can obtain this result on your calculator.

Example 3 (Finding the Adjacent Side)

Find the length of the adjacent side, marked x in the diagram.



Note that

$$\theta = 70^o$$
$$A = x$$
$$O = 10.$$

Using $\tan\theta = \frac{O}{A}$

gives $\tan 70^o = \frac{10}{x}.$

Multiplying both sides by x gives,

$$x \tan 70^o = 10$$

and dividing both sides by $\tan 70^o$ gives,

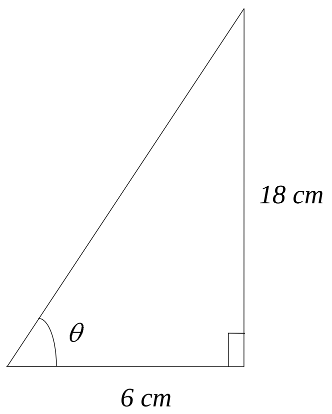
$$x = \frac{10}{\tan 70^o} = \underline{\underline{3.6 \text{ cm.}}}$$

Check that you can obtain this result.

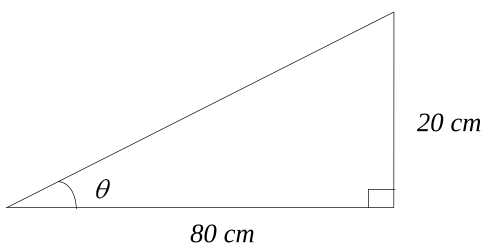
Exercise

1. Find the angle marked θ in each triangle below.

(a)

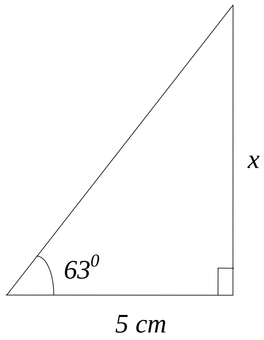


(b)

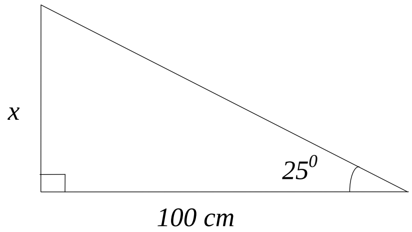


2. Find the lengths of the opposite sides marked *x* in each triangle below.

(a)

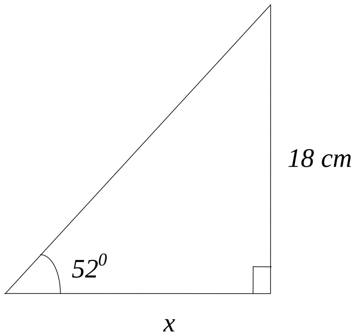


(b)

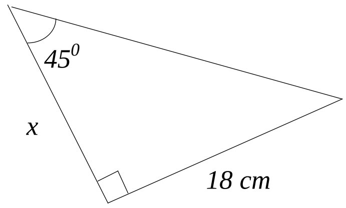


3. Find the length of the adjacent sides marked *x* in each diagram below.

(a)

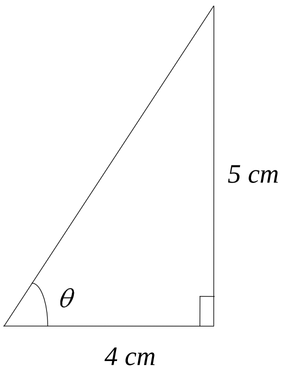


(b)

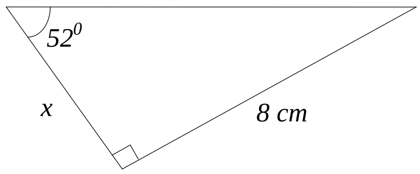


4. Find the unknown angle or length in each triangle given below.

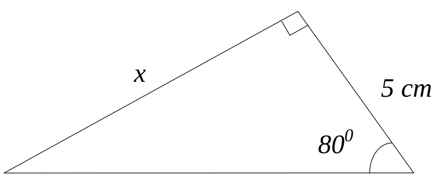
(a)



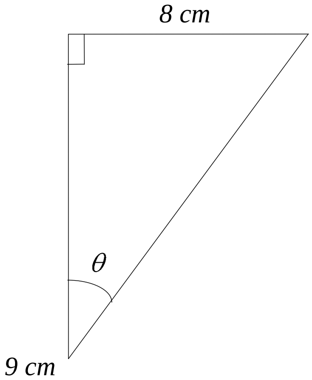
(b)



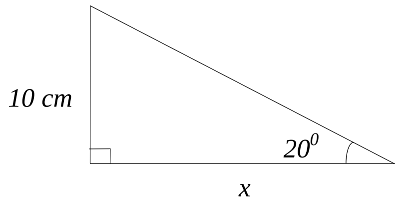
(c)



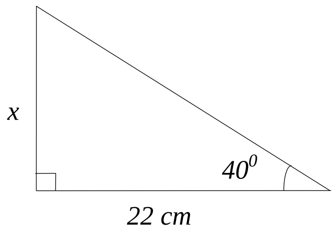
(d)



(e)



(f)



Solutions

1\. (a) $\tan\theta = \frac{18}{6}$

$= 3$

$\theta = 71.6^\circ$
(Use $\tan^{-1}$ on calculator)

(b) $\tan\theta = \frac{20}{80}$

$= 0.25$

$\theta = 14.0^\circ$
(Use $\tan^{-1}$ on calculator).

2\. (a) $\tan 63^\circ = \frac{x}{5}$

$x = 5 \times \tan 63^\circ$

$= 9.8$ cm

(b) $\tan 25^\circ = \frac{x}{100}$

$x = 100 \times \tan 25^\circ$

$= 46.6$ cm

3\. (a) $\tan 52^\circ = \frac{18}{x}$

$x = \frac{18}{\tan 52^\circ}$

$= 14.1$ cm

(b) $\tan 45^\circ = \frac{18}{x}$

$x = \frac{18}{\tan 45^\circ}$

$= 18$ cm.

4\. (a) $\tan\theta = \frac{5}{4}$

$= 1.25$

$\theta = 51.3^\circ$

(Use $\tan^{-1}$ on calculator)

(b) $\tan 52^\circ = \frac{8}{x}$

$x = \frac{8}{\tan 52^\circ}$

$= 6.3$ cm

(c) $\tan 80^\circ = \frac{x}{5}$

$x = 5 \times \tan 80^\circ$

$= 28.4$ cm

(d) $\tan\theta = \frac{8}{9}$

$= 0.8889$

$\theta = 41.6^\circ$

(Use $\tan^{-1}$ on calculator).

(e) $\tan 20^\circ = \frac{10}{x}$

$x = \frac{10}{\tan 20^\circ}$

$= 27.5$ cm

(f) $\tan 40^\circ = \frac{x}{22}$

$x = 22 \times \tan 40^\circ$

$= 18.5$ cm.

Mixed Problems

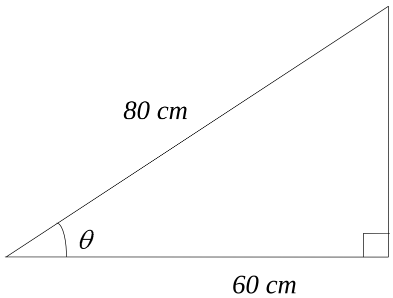
The three relationships,

$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{O}{A}$$

can be used to solve different problems. When faced with a problem it is important to decide which sides are involved in the problem. If the opposite and the adjacent are involved then you need to use tan. The following examples show how to select whether to use sin, cos or tan.

Example 1

Find the angle marked θ in the triangle shown in the diagram .



The sides involved here are the hypotenuse and the adjacent, with $H = 80$ and $A = 60$.

So we need the relationship that involves A and H. This is,

$$\cos\theta = \frac{A}{H}$$

which gives

$$\cos\theta = \frac{60}{80}$$

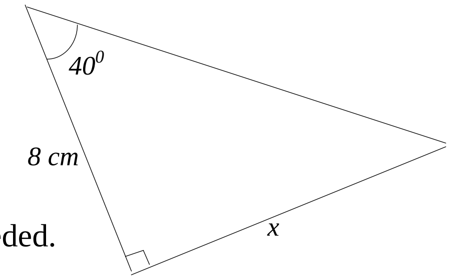
$$= 0.75.$$

So using $\cos^{-1}$ gives,

$$\underline{\underline{\theta = 41.4^{\circ}}}.$$

Example 2

Find the length marked x in the triangle.



Here the opposite and the adjacent are involved, with $A = 8$ and $O = x$.

So the relationship that involves A and O is needed.

This is;

$$\tan\theta = \frac{O}{A}$$

which gives

$$\tan 40^\circ = \frac{x}{8}$$

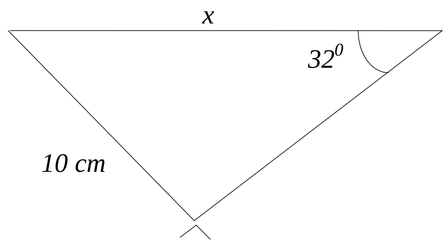
so that

$$x = 8 \times \tan 40^\circ$$

$$\underline{\underline{x = 6.7}}$$

Example 3

Find the length marked x in the triangle.



Here the opposite and hypotenuse are involved with $O = 10$ and $H = x$.

So sin is needed as this involves O and H.

Using

$$\sin\theta = \frac{O}{H}$$

gives

$$\sin 32^\circ = \frac{10}{x}.$$

Rearranging gives,

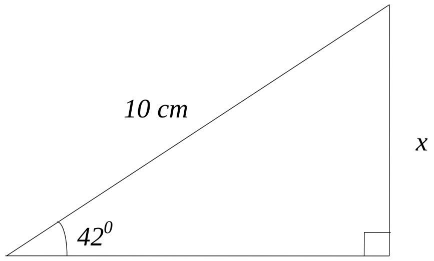
$$x = \frac{10}{\sin 32^\circ}$$

$$\underline{\underline{x = 18.9 \text{ cm.}}}$$

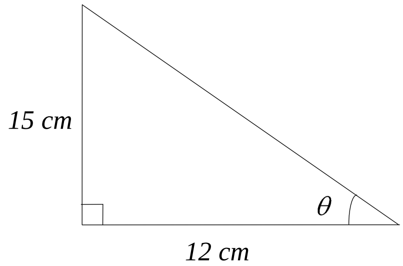
Exercise

1. For each triangle below find the angle or length marked.

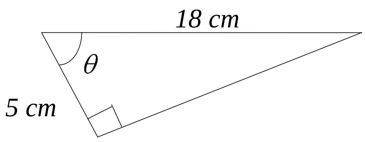
(a)



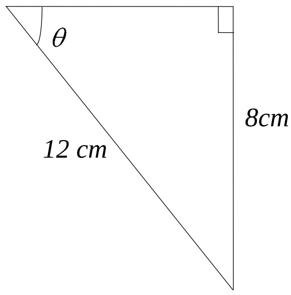
(b)



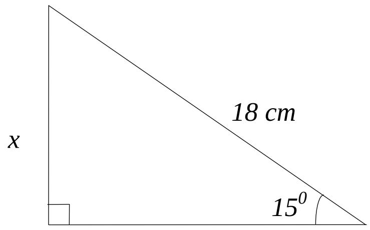
(c)



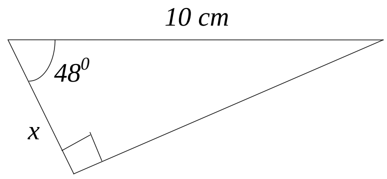
(d)



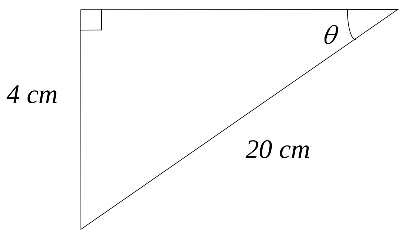
(e)



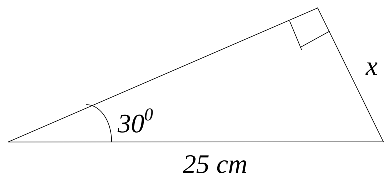
(f)



(g)



(h)



Solutions

1. (a) $\sin 42^{o} = \dfrac{x}{10}$

$x = 10 \sin 42^{o}$

$= 6.7$ cm

(b) $\tan\theta = \dfrac{15}{12}$

$= 1.25$

$\theta = 51.3^{o}$
(Use $\tan^{-1}$ on calculator)

(c) $\cos\theta = \dfrac{5}{18}$

$= 0.2778$

$\theta = 73.9^{o}$

(d) $\sin\theta = \dfrac{8}{12}$

$= 0.6667$

$\theta = 41.8^{o}$

(e) $\sin 15^{o} = \dfrac{x}{18}$

$x = 18 \times \sin 15^{o}$

$= 4.7$ cm

(f) $\cos 48^{o} = \dfrac{x}{10}$

$x = 10 \cos 48^{o}$

$= 6.7$ cm

(g) $\sin\theta = \dfrac{4}{20}$

$= 0.2$

$\theta = 11.5$ cm

(h) $\sin 30^{o} = \dfrac{x}{25}$

$x = 25 \sin 30^{o}$

$= 12.5$ cm.

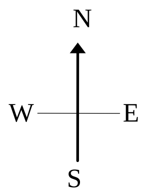
Applications

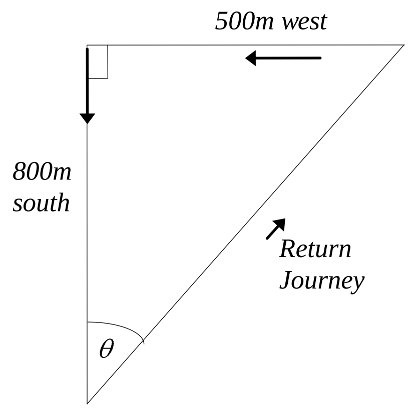
When trying to solve any problem with simple trigonometry the first key step is always to represent the information on a right angled triangle. The appropriate technique from the earlier section of this booklet can then be selected and applied.

Example 1

A competitor at an orienteering contest runs 500 m due west and then runs 800 m due south. Find the bearing on which the competitor should run to get back to her starting point.

Solution





First represent the information on a diagram. The bearing is represented by the angle θ, the angle between north and the direction of travel .

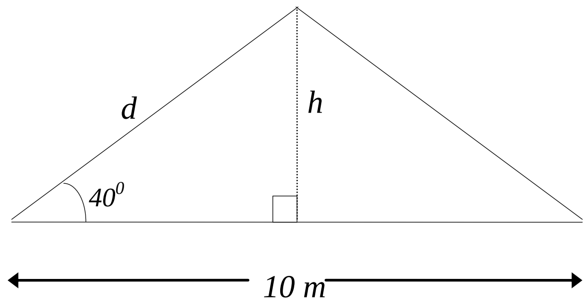
To find θ use;

$$\tan\theta = \frac{O}{A}$$
$$= \frac{500}{800}$$
$$= 0.625$$
$$\underline{\underline{\theta = 32^o}}.$$

Example 2

A roof is to span a house of width 10m and the roof must be at an angle of 40^o to the horizontal. Find the height of the peak of the roof to the heighest point.

Solution



First represent the information on a diagram. Note that in this problem we will work in half of the bigger triangle. The height is the length labelled *h* and the required distance is labelled *d*.

To find h use

$$\tan\theta = \frac{O}{A} \qquad \cos\theta = \frac{A}{H}$$

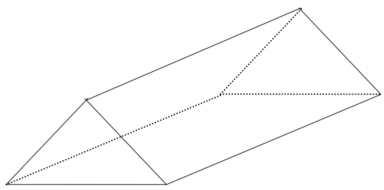
$$\tan 40^o = \frac{h}{5} \qquad \cos 40^o = \frac{5}{d}$$

$h = 5 \times \tan 40^{\circ}$ $\qquad$ $d = \dfrac{5}{\cos 40^{\circ}}$

$\underline{\underline{h = 4.20\text{m}}}$ $\qquad$ $\underline{\underline{d = 6.53\text{m}}}$.

Exercise

1. A prism has a base width of 8cm and a height of 6cm. Find the angle between the base and the sloping sides.



2. A guy rope of length 2m is attached to a vertical tent pole of height 1.8m. Find the angle between the pole and the guy rope.

3. A geographer measures a distance of 200m up a slope on a hill. He estimates that he has risen 10m. Find the angle between the slope and the horizontal.

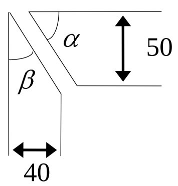
4. A roof spans a house of width 8m. The length of the sloping roof is 6m on each side. Find the angle between the roof and the horizontal. Also find the difference between the lowest and highest point on the roof.

5. A ladder of length 3m leans against a wall, so that the end of the ladder is 1.5m from the base of the wall. Find the angle between the ladder and the ground and the height of the top of the ladder.

6. A competitor in an orienteering contest runs 400m east and then runs due north until he is 600m from his starting point. Find the bearing on which a competitor could have run to get directly to this point.

7. A washing line of length 12m hangs so that it forms a v shape with its lowest point is 2m lower than each end. Find the angle between the line and the horizontal.

8. A picture is made by joining wood of different widths. If one width is 50mm and the other 40mm, find the angles α and β shown on the diagram.

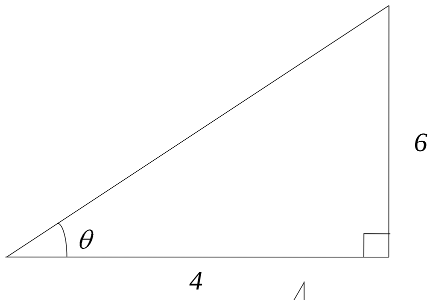


9. The jib of a mobile crane has length 10m and is inclined at an angle of 18° to the vertical.

   (a) Find the height of the end of the jib.
   (b) What angle should there be between the jib and the vertical for a height of 7m?

Solutions

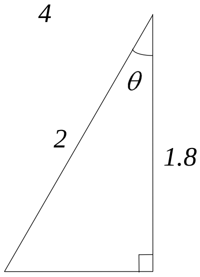
1.



$\tan\theta = \frac{6}{4}$

$\theta = 56.3^{o}$

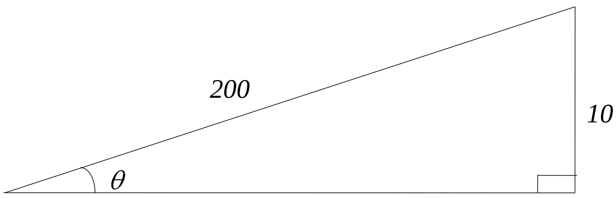
2.



$\cos\theta = \frac{1.8}{2}$

$\theta = 25.8^{o}$

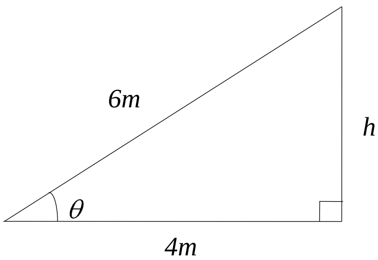
3.



$\sin\theta = \frac{10}{200}$

$\theta = 2.9^{o}$

4.



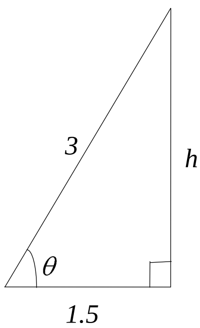
$\cos\theta = \frac{4}{6}$

$\theta = 48.2^{o}$

$h = 6 \times \sin 48.2^{o}$

$= 4.47$m.

5.



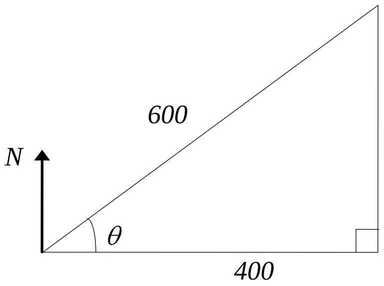
$\cos\theta = \frac{1.5}{3}$

$\theta = 60^{o}$

$h = 3 \sin 60^{o}$

$= 2.60$m.

6.



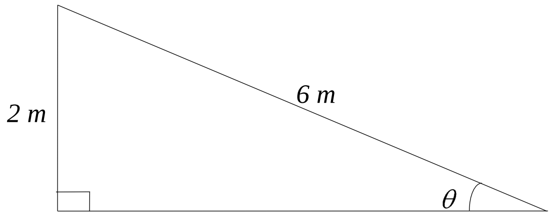
$\cos\theta = \dfrac{400}{600}$

$\theta = 48.2^o$

Bearing = $90 - 48.2^o$

$= 41.8^o$.

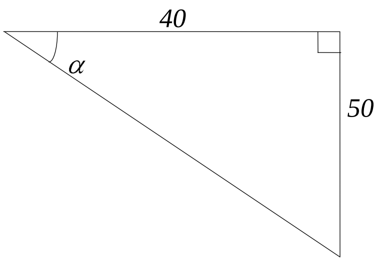
7.



$\sin\theta = \dfrac{2}{6}$

$\theta = 19.5^o$

8.



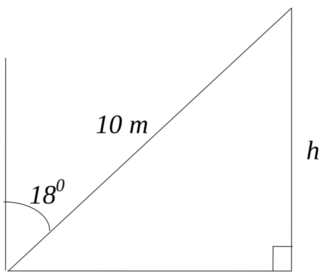
$\tan\alpha = \dfrac{50}{40}$

$\alpha = 51.3^o$

$\beta = 90 - 51.3$

$= 38.7^o$

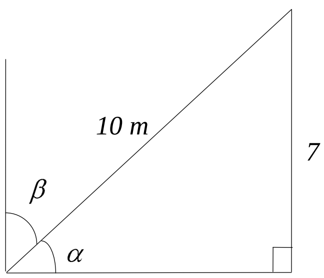
9. (a)



$\sin(72^o) = \dfrac{h}{10}$

$h = 9.51$m

(b)



$\sin\alpha = \dfrac{7}{10}$

$\alpha = 44.4^o$

$\beta = 90 - 44.4^o$

$= 45.6^o$

Mathematics Support Series

Working with Fractions

Ted Graham and Pat Patel
The Centre for Teaching Mathematics
University of Plymouth

Introduction

Fractions can arise in many situations. Weights and lengths, in spite of metrication, are still often worked with in fractions. Probabilities that you may meet in other courses are often expressed in fractions. In mathematics when a fraction such as $\frac{1}{13}$ is converted to a decimal you obtain an approximate result and lose a degree of accuracy due to the rounding that takes place. For this reason it can be an advantage to work with fractions rather than a decimal form of the numbers.

If you are to work with algebraic fractions such as $\frac{a}{b}$ or $\frac{6x}{5x+3}$, it will be to your advantage if you are familiar with the usage of ordinary fractions. This booklet will not cover algebraic fractions, but its contents will help you a great deal if you encounter these in other aspects of your mathematics.

This booklet will cover:

(i) Equivalent Fractions.
(ii) Mixed Numbers and Improper Fractions.
(iii) Addition and Subtraction of Fractions.
(iv) Multiplication of Fractions.
(v) Division with Fractions.

Equivalent Fractions

It is possible to write fractions in different but equivalent forms. For example $\frac{2}{4}$ is the same as $\frac{1}{2}$, or $\frac{8}{10}$ is the same as $\frac{4}{5}$. The smaller the numbers involved in a fraction the easier it is to work with them or to use them. Fractions are normally expressed in their simplest forms, such as $\frac{1}{2}$. It would be very unusual to hear someone speak about or use two quarters $\left(\frac{2}{4}\right)$ instead. It is important to be able to reduce fractions to their simplest form. It is also important to be able to write fractions in a variety of different ways. This is particularly important in the processes of addition and subtraction.

To simplify a fraction the top (or numerator) of the fraction and the bottom (or denominator) of the fraction should both be divided by the same number.

Example 1

To simplify $\frac{8}{60}$ find the biggest number that both the top and bottom of the fraction can be divided by. In this case both numbers can be divided by 4 to give:

$$\frac{8}{60} = \frac{2}{15}$$

Alternatively you can divide the top and bottom of the fraction in stages, for example,

$$\frac{8}{60} = \frac{4}{30} = \frac{2}{15}$$

Here the numbers are divided by 2 at each stage.

Example 2

To simplify $\frac{18}{45}$ note that both numbers can be divided by 9 to give $\frac{2}{5}$.

Example 3

To simplify $\frac{108}{32}$ note that both numbers can be divided by 4 to give $\frac{27}{8}$.

Example 4

To convert $\frac{3}{8}$ into a fraction with a bottom (or denominator) of 96, multiply both the top and the bottom of the fraction by 12, to give $\frac{36}{96}$.

Exercise

1. Write each fraction in its simplest form.

(a) $\frac{8}{12}$ (f) $\frac{33}{81}$ (k) $\frac{16}{84}$

(b) $\frac{9}{99}$ (g) $\frac{72}{100}$ (l) $\frac{15}{75}$

(c) $\frac{44}{121}$ (h) $\frac{52}{96}$ (m) $\frac{17}{119}$

(d) $\frac{16}{52}$ (i) $\frac{13}{18}$ (n) $\frac{27}{69}$

(e) $\frac{21}{56}$ (j) $\frac{22}{54}$ (o) $\frac{40}{75}$

2. (a) $\frac{6}{7} = \frac{\square}{21}$ (c) $\frac{2}{3} = \frac{\square}{18}$ (e) $\frac{11}{21} = \frac{\square}{63}$

(b) $\frac{4}{5} = \frac{\square}{80}$ (d) $\frac{4}{7} = \frac{\square}{77}$ (f) $\frac{12}{40} = \frac{48}{\square}$

Solutions

1. (a) $\frac{8}{12} = \frac{2}{3}$ (dividing by 4)

(b) $\frac{9}{99} = \frac{1}{11}$ (dividing by 11)

(c) $\frac{44}{121} = \frac{4}{11}$ (dividing by 4)

(d) $\frac{16}{52} = \frac{4}{13}$ (dividing by 4)

(e) $\frac{21}{56} = \frac{3}{8}$ (dividing by 7)

(f) $\frac{33}{81} = \frac{11}{27}$ (dividing by 3)

(i) $\frac{13}{18}$ does not simply

(j) $\frac{22}{54} = \frac{11}{27}$ (dividing by 2)

(k) $\frac{16}{84} = \frac{4}{21}$ (dividing by 4)

(l) $\frac{15}{75} = \frac{1}{5}$ (dividing by 15)

(m) $\frac{17}{119} = \frac{1}{7}$ (dividing by 17)

(n) $\frac{27}{69} = \frac{9}{23}$ (dividing by 3)

(g) $\frac{72}{100} = \frac{18}{25}$ (dividing by 4)

(o) $\frac{40}{75} = \frac{8}{15}$ (dividing by 5)

(h) $\frac{52}{96} = \frac{13}{24}$ (dividing by 4)

2. (a) $\frac{6}{7} = \frac{18}{21}$ (multiply by 3)

(d) $\frac{4}{7} = \frac{44}{77}$ (multiply by 11)

(b) $\frac{4}{5} = \frac{64}{80}$ (multiply by 16)

(e) $\frac{11}{21} = \frac{33}{63}$ (multiply by 3)

(c) $\frac{2}{3} = \frac{12}{18}$ (multiply by 6)

(f) $\frac{12}{40} = \frac{48}{160}$ (multiply by (4)

Improper Fractions and Mixed Numbers

An improper fraction is one where the top (or numerator) is greater than the bottom (or denominator), for example $\frac{16}{7}$, $\frac{12}{3}$ and $\frac{6}{5}$ are all improper fractions.

A mixed number contains a whole number and a fraction. For example $3\frac{1}{4}$ and $5\frac{6}{7}$ are both mixed numbers.

To be able to convert between improper fractions and mixed numbers is an important skill.

Example 1

Convert $5\frac{1}{4}$ to an improper fraction.

Solution

To do this, multiply the 5 by 4 and add 1, to get the top (or numerator) the improper fraction. The bottom or denominator stays the same. So in this case $5 \times 4 = 20$ and adding 1 gives 21, so that the fraction is $\frac{21}{4}$.

Example 2

Convert $3\frac{4}{9}$ to an improper fraction.

Solution

The bottom or (denominator) stays as 9. To find the numerator, multiply the 3×9 to give 27, and add 4 to give 31. So the improper fraction is $\frac{31}{9}$.

Example 3

Convert $\frac{19}{4}$ to a mixed number.

Solution

First divide 19 by 4. This gives 4 with a remainder of 3. So the mixed number is $4\frac{3}{4}$.

Example 4

Convert $\frac{118}{9}$ to a mixed number.

Solution

Dividing 118 by 9 gives 13 and a remainder of 1, as $13 \times 9 = 117$. So the mixed number will be $13\frac{1}{9}$.

Exercise

1. Convert each mixed number below into an improper fraction.

(a)	$4\frac{1}{5}$	(d)	$5\frac{2}{21}$	(g)	$5\frac{16}{19}$
(b)	$3\frac{7}{8}$	(e)	$3\frac{4}{19}$	(h)	$4\frac{3}{7}$
(c)	$5\frac{4}{7}$	(f)	$1\frac{2}{17}$	(i)	$5\frac{3}{16}$

2. Convert each improper function into a mixed number.

(a)	$\frac{5}{2}$	(d)	$\frac{16}{5}$	(g)	$\frac{100}{62}$
(b)	$\frac{7}{3}$	(e)	$\frac{107}{3}$	(h)	$\frac{58}{9}$
(c)	$\frac{10}{4}$	(f)	$\frac{19}{5}$	(i)	$\frac{46}{11}$

Solutions

1. (a) $4\times 5+1=21$
so $4\frac{1}{5}=\frac{21}{5}$

(b) $3\times 8+7=31$
so $3\frac{7}{8}=\frac{31}{8}$

(c) $5\times 7+4=39$
so $5\frac{4}{7}=\frac{39}{7}$

(d) $4\times 21+2=107$
so $5\frac{2}{21}=\frac{107}{21}$

(e) $3\times 19+4=61$
so $3\frac{4}{19}=\frac{61}{19}$

(f) $1\times 17+2=19$
so $1\frac{2}{17}=\frac{19}{17}$

(g) $5\times 19+16=111$
so $5\frac{16}{19}=\frac{111}{19}$

(h) $4\times 7+3=31$
so $4\frac{3}{7}=\frac{31}{7}$

(i) $5\times 16+3=83$
so $5\frac{3}{16}=\frac{83}{16}$

2. (a) Dividing 5 by 2 give 2 remainder 1, so $\frac{5}{2}=2\frac{1}{2}$.

(b) Dividing 7 by 3 gives 2 remainder 1, so $\frac{7}{3}=2\frac{1}{3}$.

(c) Dividing 10 by 4 gives 2 remainder 2, so $\frac{10}{4}=2\frac{2}{4}=2\frac{1}{2}$.

(d) Dividing 16 by 5 gives 3 remainder 1, so $\frac{16}{5}=3\frac{1}{5}$.

(e) Dividing 107 by 3 gives 35 remainder 2, so $\frac{107}{3}=35\frac{2}{3}$.

(f) Dividing 19 by 5 gives 3 remainder 4, so $\frac{19}{5}=3\frac{4}{5}$.

(g) Dividing 100 by 62 gives 1 remainder 38, so $\frac{100}{62}=1\frac{38}{62}=1\frac{19}{31}$..

(h) Dividing 58 by 9 gives 6 remainder 4, so $\frac{58}{9}=6\frac{4}{9}$.

(i) Dividing 46 by 11 gives 4 remainder 2, so $\frac{46}{11}=4\frac{2}{11}$.

Addition and Subtraction of Fractions

It is a very simple matter to add fractions which have the same number on the bottom or denominator. For example,

$$\frac{3}{11}+\frac{2}{11}=\frac{5}{11}.$$

Note that the numbers on the top are added together, but the numbers underneath are not. The same is true in subtraction, for example,

$$\frac{5}{8}-\frac{2}{8}=\frac{3}{8}.$$

Here the 2 is subtracted from the 5, but again the bottom numbers do not change in any way.

When faced with a problem such as

$$\frac{3}{8}+\frac{4}{5}$$

it is necessary to change the fractions so that the two numbers underneath are the same. This number is often called the common denominator. In this case it will be 40 as this is the smallest number that can be divided by both 5 and 8. Often, but not always, the common denominator can be found by multiplying together the two denominators. So the original problem can be re-written as,

$$\frac{3}{8}+\frac{4}{5}=\frac{15}{40}+\frac{32}{40}.$$

Note that the top of each fraction has been multiplied by the same number as the bottom of each fraction. For the first fraction the 8 was multiplied by 5 to give 40, so the 3 was also multiplied by 5 to give 15. For the second fraction, the 5 was multiplied by 8 to give 40, so the 4 was also multiplied by 8 to give 32.

Now the fractions can be added,

$$\underline{\underline{\frac{15}{40}+\frac{32}{40}=\frac{47}{40}=1\frac{7}{40}}}.$$

The same approach can be used in problems that involve subtraction.

Example 1

Find $\frac{4}{5}+\frac{2}{3}$.

Solution

First the common denominator must be found. In this case it is 15, which is the smallest number that both 5 and 3 divide exactly. So multiply the top and bottom of the first fraction by 3 to give $\frac{12}{15}$ and then multiply the top and bottom of the second fraction by 5 to give $\frac{10}{15}$. Now the two fractions can be added together:

$$\frac{4}{5}+\frac{2}{3}=\frac{12}{15}+\frac{10}{15}$$

$$=\frac{22}{15}$$

$$=1\frac{7}{15}.$$

Example 2

Find $\frac{5}{6}+\frac{3}{10}$.

Solution

Here the common denominator is 30, because this is the smallest number that both 6 and 10 divide into. The top and bottom of the first fraction must be multiplied by 5 to give $\frac{25}{30}$, and the top and bottom of the second fraction must be multiplied by 3 to give $\frac{9}{30}$. So the addition can then take place

$$\frac{5}{6}+\frac{3}{10}=\frac{25}{30}+\frac{9}{30}$$

$$=\frac{34}{30}$$

$$=1\frac{4}{30}$$

$$=1\frac{2}{15}.$$

Example 3

Find $\frac{3}{7}+\frac{2}{9}$.

Solution

Here the common denominator is 63.

Explain why the addition becomes

$$\frac{3}{7}+\frac{2}{9}=\frac{27}{63}+\frac{14}{63}$$

$$=\underline{\underline{\frac{41}{63}}}.$$

Exercise

1. Find the answer to each problem below.

(a) $\frac{3}{5}+\frac{1}{5}=$

(b) $\frac{3}{8}+\frac{1}{8}=$

(c) $\frac{5}{7}+\frac{1}{7}=$

(d) $\frac{8}{13}-\frac{5}{13}=$

(e) $\frac{7}{9}-\frac{4}{9}=$

(f) $\frac{7}{10}-\frac{3}{10}=$

2. Complete the blanks in each case below:

(a) $\frac{2}{5}+\frac{3}{7}=\frac{\square}{35}+\frac{15}{35}=\frac{\square}{35}$

(b) $\frac{1}{5}+\frac{1}{6}=\frac{\square}{30}+\frac{\square}{30}=\frac{\square}{\square}$

(c) $\frac{4}{7}+\frac{2}{3}=\frac{\square}{21}+\frac{\square}{21}=\frac{\square}{\square}$

(d) $\frac{1}{2}+\frac{1}{4}=\frac{\square}{4}+\frac{1}{4}=\frac{\square}{4}$

(e) $\frac{3}{16}+\frac{5}{8}=\frac{\square}{16}+\frac{\square}{16}=\frac{\square}{\square}$

(f) $\frac{3}{5}+\frac{7}{12}=\frac{\square}{\square}+\frac{\square}{\square}=\frac{\square}{\square}$

3. Add together the fractions given each case.

(a) $\frac{1}{6}+\frac{3}{8}=$

(e) $\frac{3}{7}+\frac{5}{8}=$

(b) $\frac{5}{7}+\frac{2}{5}=$ (f) $\frac{1}{2}+\frac{2}{3}=$

(c) $\frac{1}{8}+\frac{3}{32}=$ (g) $\frac{1}{7}+\frac{1}{10}=$

(d) $\frac{1}{10}+\frac{1}{3}=$ (h) $\frac{5}{8}+\frac{4}{3}=$

4. Use the technique of finding a common denominator to carry out the following subtractions.

(a) $\frac{4}{7}-\frac{1}{2}=$ (d) $\frac{3}{4}-\frac{2}{3}=$

(b) $\frac{6}{11}-\frac{1}{4}=$ (e) $\frac{5}{8}-\frac{5}{12}=$

(c) $\frac{2}{3}-\frac{1}{6}=$ (f) $\frac{11}{12}-\frac{3}{8}=$

5. A shopper buys $1\frac{1}{4}$ kg of golden delicious apples and $1\frac{1}{3}$ kg of coxes apples. Find the total weight of the apples bought.

6. A keen gardener owns a garden that has an area of $\frac{3}{5}$ acre. He is able to buy some adjoining land with an area of $\frac{3}{8}$ acre. Find the total area of his new garden.

7. A picture has dimensions $10\frac{3}{4}''$ by $12\frac{2}{5}''$. A frame is put round the picture. The width of the frame is $1\frac{1}{3}''$.

(a) Find the perimeter of the original picture.
(b) Find the dimensions of the framed picture.
(c) Find the perimeter of the framed picture.

8. A successful company claims to have had profits of: £$\frac{3}{4}$ million, £$1\frac{1}{5}$ million and £$1\frac{1}{2}$ million, in three consecutive years. Find the total profit over the 3 year period.

Solutions

1. (a) $\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$.

(b) $\frac{3}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$.

(c) $\frac{5}{7}+\frac{1}{7}=\frac{6}{7}$.

(d) $\frac{8}{13}-\frac{5}{13}=\frac{3}{13}$.

(e) $\frac{7}{9}-\frac{4}{9}=\frac{3}{9}=\frac{1}{3}$.

(f) $\frac{7}{10}-\frac{3}{10}=\frac{4}{10}=\frac{2}{5}$.

2. (a) $\frac{2}{5}+\frac{3}{7}=\frac{14}{35}+\frac{15}{35}=\frac{29}{35}$.

(b) $\frac{1}{5}+\frac{1}{6}=\frac{6}{30}+\frac{5}{30}=\frac{11}{30}$.

(c) $\frac{4}{7}+\frac{2}{3}=\frac{12}{21}+\frac{14}{21}=\frac{26}{21}$.

(d) $\frac{1}{2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$.

(e) $\frac{3}{16}+\frac{5}{8}=\frac{3}{16}+\frac{10}{16}=\frac{13}{16}$.

(f) $\frac{3}{5}+\frac{7}{12}=\frac{36}{60}+\frac{35}{60}=\frac{71}{60}=1\frac{11}{60}$.

3. (a) $\frac{1}{6}+\frac{3}{8}=\frac{4}{24}+\frac{9}{24}=\frac{13}{24}$.

(b) $\frac{5}{7}+\frac{2}{5}=\frac{25}{35}+\frac{14}{35}=\frac{39}{35}=1\frac{4}{35}$.

(c) $\frac{1}{8}+\frac{3}{32}=\frac{4}{32}+\frac{3}{32}=\frac{7}{32}$.

(d) $\frac{1}{10}+\frac{1}{3}=\frac{3}{30}+\frac{10}{30}=\frac{13}{30}$.

(e) $\frac{3}{7}+\frac{5}{8}=\frac{24}{56}+\frac{35}{56}=\frac{59}{56}=1\frac{3}{56}.$

(f) $\frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}=1\frac{1}{6}.$

(g) $\frac{1}{7}+\frac{1}{10}=\frac{10}{70}+\frac{7}{70}=\frac{17}{70}.$

(h) $\frac{5}{8}+\frac{4}{3}=\frac{15}{24}+\frac{32}{24}=\frac{47}{24}=1\frac{23}{24}.$

4. (a) $\frac{4}{7}-\frac{1}{2}=\frac{8}{14}-\frac{7}{14}=\frac{1}{14}.$

(b) $\frac{6}{11}-\frac{1}{4}=\frac{24}{44}-\frac{11}{44}=\frac{13}{44}.$

(c) $\frac{2}{3}-\frac{1}{6}=\frac{4}{6}-\frac{1}{6}=\frac{3}{6}=\frac{1}{2}.$

(d) $\frac{3}{4}-\frac{2}{3}=\frac{9}{12}-\frac{8}{12}=\frac{1}{12}.$

(e) $\frac{5}{8}-\frac{5}{12}=\frac{15}{24}-\frac{10}{24}=\frac{5}{24}.$

(f) $\frac{11}{12}-\frac{3}{8}=\frac{22}{24}-\frac{9}{24}=\frac{13}{24}.$

5. $1\frac{1}{4}+1\frac{1}{3}=1\frac{3}{12}+1\frac{4}{12}=2\frac{7}{12}$ kg.

6. $\frac{3}{5}+\frac{3}{8}=\frac{24}{40}+\frac{15}{40}=\frac{39}{40}$ acre.

7. (a) $10\frac{3}{4}+10\frac{3}{4}+12\frac{2}{5}+12\frac{2}{5}=20\frac{6}{4}+24\frac{4}{5}$

$=21\frac{1}{2}+24\frac{4}{5}$

$=21\frac{5}{10}+24\frac{8}{10}$

$=45\frac{13}{10}$

$=46\frac{3}{10}.$

(b) $10\frac{3}{4}+1\frac{1}{3}+1\frac{1}{3}=10\frac{3}{4}+2\frac{2}{3}=10\frac{9}{12}+2\frac{8}{12}$

$$= 12\frac{17}{12}$$

$$= 13\frac{5}{12}.$$

$$12\frac{2}{5}+1\frac{1}{3}+1\frac{1}{3} = 12\frac{2}{5} + 2\frac{2}{3} = 12\frac{6}{15}+2\frac{10}{15}$$

$$= 14\frac{16}{15}$$

$$= 15\frac{1}{15}.$$

(c) $$13\frac{5}{12}+13\frac{5}{12}+15\frac{1}{15}+15\frac{1}{15}=26\frac{5}{6}+30\frac{2}{15}$$

$$= 26\frac{25}{30}+30\frac{4}{30}$$

$$= 56\frac{29}{30}.$$

8. $$\frac{3}{4}+1\frac{1}{5}+1\frac{1}{2}=\frac{15}{20}+1\frac{4}{20}+1\frac{10}{20}=2\frac{29}{20}=3\frac{9}{20}.$$

Multiplication with Fractions

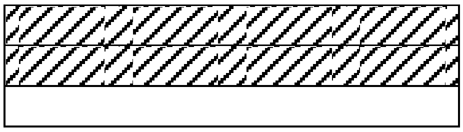
First we will consider multiplying a fraction by a whole number. Here the approach is simply to multiply the top of the fraction by the whole number.

Example 1

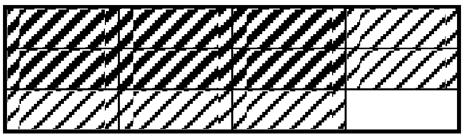
(a) $4 \times \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}$.

(b) $5 \times \frac{3}{8} = \frac{15}{8} = 1\frac{7}{8}$.

When multiplying a fraction by a fraction the approach is similar. Consider $\frac{3}{4} \times \frac{2}{3}$. The diagram shows a rectangle that has been split into thirds, and $\frac{2}{3}$ have been shaded to correspond to the $\frac{2}{3}$.



You can think of $\frac{3}{4} \times \frac{2}{3}$ as asking for $\frac{3}{4}$ of $\frac{2}{3}$. This is shown in the diagram below.



The heavily shaded area is the result and corresponds to $\frac{6}{12}$ of the original area. So

$$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}.$$

Note that this result could be obtained by multiplying the top two numbers together and the bottom two numbers together.

$$\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2}.$$

Example 1

Find $\frac{4}{5} \times \frac{3}{8}$.

Solution

Multiply the top numbers together and then multiply the bottom numbers together. This gives:

$$\frac{4}{5} \times \frac{3}{8} = \frac{4 \times 3}{5 \times 8} = \frac{12}{40} = \frac{3}{10}.$$

Example 2

Find $1\frac{1}{4} \times 2\frac{1}{5}$.

Solution

First it is necessary to convert both mixed numbers to top heavy fractions. This gives:

$$1\frac{1}{4} \times 2\frac{1}{5} = \frac{5}{4} \times \frac{11}{5}.$$

Now the top numbers can be multiplied together and then the bottom numbers can be multiplied to give:

$$\frac{5}{4} \times \frac{11}{5} = \frac{55}{20} = \frac{11}{4}.$$

Exercises

1. Carry out each multiplication below.

(a) $5 \times \frac{3}{8} =$

(b) $4 \times \frac{3}{5} =$

(c) $6 \times \frac{2}{3} =$

(d) $5 \times \frac{3}{7} =$

(e) $7 \times 1\frac{1}{4} =$

(f) $8 \times 1\frac{2}{5} =$

2. Carry out each multiplication below.

(a) $\frac{3}{4} \times \frac{5}{7} =$

(e) $\frac{4}{5} \times 1\frac{3}{7} =$

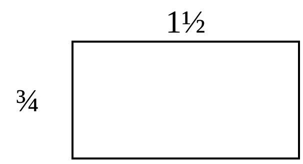
(b) $\frac{1}{5}\times\frac{7}{8}=$ (f) $1\frac{2}{5}\times 1\frac{4}{9}=$

(c) $\frac{4}{5}\times\frac{1}{12}=$ (g) $1\frac{3}{4}\times 1\frac{1}{12}=$

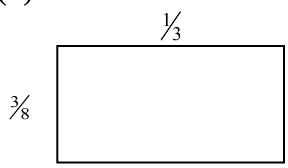
(d) $\frac{3}{7}\times\frac{9}{10}=$ (h) $2\frac{1}{2}\times 3\frac{1}{4}=$

3. Find the area of each rectangle below:

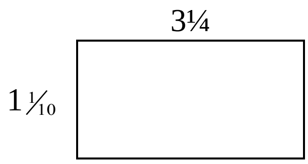
(a)



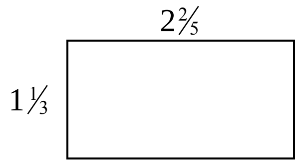
(b)



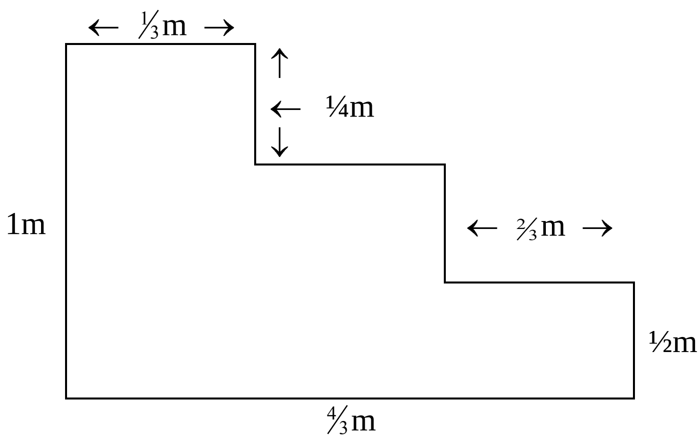
(c)



(d)



4. Find the area of the shape below in m^2.



Solutions

1. (a) $5 \times \frac{3}{8} = \frac{15}{8} = 1\frac{7}{8}$.

(b) $4 \times \frac{3}{5} = \frac{12}{5} = 2\frac{2}{5}$.

(c) $6 \times \frac{2}{3} = \frac{12}{3} = 4$.

(d) $5 \times \frac{3}{7} = \frac{15}{7} = 2\frac{1}{7}$.

(e) $7 \times 1\frac{1}{4} = 7 \times \frac{5}{4} = \frac{35}{4} = 8\frac{3}{4}$.

(f) $8 \times 1\frac{2}{5} = 8 \times \frac{7}{5} = \frac{56}{5} = 11\frac{2}{5}$.

2 (a) $\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$.

(b) $\frac{1}{5} \times \frac{7}{8} = \frac{1 \times 7}{5 \times 8} = \frac{7}{40}$.

(c) $\frac{4}{5} \times \frac{1}{12} = \frac{4 \times 1}{5 \times 12} = \frac{4}{60} = \frac{1}{15}$.

(d) $\frac{3}{7} \times \frac{9}{10} = \frac{3 \times 9}{7 \times 10} = \frac{27}{70}$.

(e) $\frac{4}{5} \times 1\frac{3}{7} = \frac{4}{5} \times \frac{10}{7} = 1\frac{5}{35} = 1\frac{1}{7}$.

(f) $1\frac{2}{5} \times 1\frac{4}{9} = \frac{7}{5} \times \frac{13}{9} = \frac{91}{45} = 2\frac{1}{45}$.

(g) $1\frac{3}{4} \times 1\frac{1}{12} = \frac{7}{4} \times \frac{13}{12} = \frac{91}{48} = 1\frac{43}{48}$.

(h) $2\frac{1}{2} \times 3\frac{1}{4} = \frac{5}{2} \times \frac{13}{4} = \frac{65}{8} = 8\frac{1}{8}$.

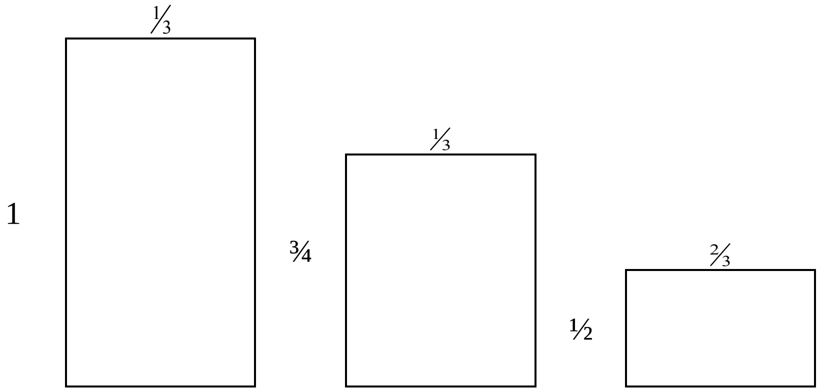
3. (a) $\frac{3}{4} \times 1\frac{1}{2} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$.

(b) $\frac{1}{3} \times \frac{3}{8} = \frac{3}{24} = \frac{1}{8}$.

(c) $3\frac{1}{4} \times 1\frac{1}{10} = \frac{13}{4} \times \frac{11}{10} = \frac{143}{40} = 3\frac{23}{40}$.

(d) $2\frac{2}{5} \times 1\frac{1}{3} = \frac{12}{5} \times \frac{4}{3} = \frac{48}{15} = 3\frac{3}{15} = 3\frac{1}{5}$.

4.



Area = $1\times\frac{1}{3}+\frac{3}{4}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{3}$

$= \frac{1}{3}+\frac{1}{4}+\frac{1}{3}$

$= \frac{2}{3}+\frac{1}{4}$

$= \frac{8}{12}+\frac{3}{12}$

$= \frac{11}{12}$ m^2.

Division with Fractions

To understand how to divide with fractions it is helpful to consider the relationship between multiplication and division. First consider

$$3 \times 4 = 12.$$

From this it is possible to deduce that

$$12 \div 4 = 3.$$

The same is true with fractions. So if we consider

$$\frac{4}{5} \times \frac{3}{7} = \frac{12}{35}$$

we could deduce that

$$\frac{12}{35} \div \frac{3}{7} = \frac{4}{5}.$$

Now also note that

$$\frac{12}{35} \times \frac{7}{3} = \frac{84}{105} = \frac{4}{5}.$$

Comparing these last two results we can see that to divide by a fraction we must turn it upside down and then multiply.

Example 1

Find $\frac{3}{8} \div \frac{5}{6}$.

Solution

To obtain the answer turn the second fraction upside down and multiply to give:

$$\frac{3}{8} \div \frac{5}{6} = \frac{3}{8} \times \frac{6}{5} = \frac{18}{40}. = \underline{\underline{\frac{9}{20}}}.$$

Example 2

Find $1\frac{3}{8} \div \frac{5}{7}$.

Solution

As with multiplication, mixed numbers must be converted to top heavy fractions. So

$$1\frac{3}{8} \div \frac{5}{7}$$

becomes,

$$\frac{11}{8} \div \frac{5}{7} = \frac{11}{8} \times \frac{7}{5} = \frac{77}{40}. = 1\frac{37}{40}$$

Exercises

1. Find

(a) $\frac{3}{4} \div \frac{1}{2} =$

(b) $\frac{6}{7} \div \frac{3}{4} =$

(c) $\frac{1}{5} \div \frac{1}{7} =$

(d) $\frac{3}{8} \div \frac{4}{5} =$

(e) $1\frac{1}{4} \div \frac{3}{8} =$

(f) $5\frac{1}{2} \div 1\frac{1}{4} =$

(g) $1\frac{1}{7} \div 2\frac{3}{8} =$

(h) $4\frac{1}{2} \div 1\frac{1}{5} =$

2. By noting that $2 = \frac{2}{1}$ etc. find;

(a) $\frac{1}{7} \div 2 =$

(c) $\frac{3}{8} \div 10 =$

(c) $6 \div \frac{3}{8} =$

(d) $7 \div 1\frac{1}{4} =$

3. A landowner leaves an estate to be divided equally between 9 grandchildren. The value of the estate was £1¾ million. Find the amount that each receives as a fraction of £ million.

Solutions

1 (a) $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{6}{4} = 1\frac{1}{2}$.

(b) $\frac{6}{7} \div \frac{3}{4} = \frac{6}{7} \times \frac{4}{3} = \frac{24}{21} = 1\frac{1}{7}$.

(c) $\frac{1}{5} \div \frac{1}{7} = \frac{1}{5} \times \frac{7}{1} = \frac{7}{5}$.

(d) $\frac{3}{8} \div \frac{4}{5} = \frac{3}{8} \times \frac{5}{4} = \frac{15}{32}$.

(e) $1\frac{1}{4} \div \frac{3}{8} = \frac{5}{4} \div \frac{3}{8} = \frac{5}{4} \times \frac{8}{3} = \frac{40}{12} = 3\frac{1}{3}$.

(f) $5\frac{1}{2} \div 1\frac{1}{4} = \frac{11}{2} \div \frac{5}{4} = \frac{11}{2} \times \frac{4}{5} = \frac{44}{10} = 4\frac{2}{5}$

(g) $1\frac{1}{7} \div 2\frac{3}{8} = \frac{8}{7} \div \frac{19}{8} = \frac{8}{7} \times \frac{8}{19} = \frac{64}{133}$.

(h) $4\frac{1}{2} \div 1\frac{1}{5} = \frac{9}{2} \div \frac{6}{5} = \frac{9}{2} \times \frac{5}{6} = \frac{45}{12} = 3\frac{3}{4}$.

2. (a) $\frac{1}{7} \div 2 = \frac{1}{7} \div \frac{2}{1} = \frac{1}{7} \times \frac{1}{2} = \frac{1}{14}$

(b) $\frac{3}{8} \div 10 = \frac{3}{8} \div \frac{10}{1} = \frac{3}{8} \times \frac{1}{10} = \frac{3}{80}$

(c) $6 \div \frac{3}{8} = \frac{6}{1} \div \frac{3}{8} = \frac{6}{1} \times \frac{8}{3} = \frac{48}{3} = 16$

(d) $7 \div 1\frac{1}{4} = \frac{7}{1} \div \frac{5}{4} = \frac{7}{1} \times \frac{4}{5} = \frac{28}{5} = 5\frac{3}{5}$

3. $1\frac{3}{4} \div 9 = \frac{7}{4} \div \frac{9}{1} = \frac{7}{4} \times \frac{1}{9} = \frac{7}{36}$

Mixed Problems

The following exercise contains a mixture of problems to test your understanding of how to work with fractions.

Exercise

1. Find

(a) $\frac{3}{8} \times 1\frac{4}{7} =$

(b) $\frac{3}{6} + \frac{2}{7} =$

(c) $1\frac{4}{5} - 1\frac{1}{3} =$

(d) $\frac{8}{17} \div 1\frac{1}{2} =$

(e) $\frac{3}{8} + \frac{2}{7} =$

(f) $5\frac{1}{2} \times 3\frac{1}{4} =$

2. In this question remember to find the contents of each bracket first.

(a) $1\frac{1}{5} \times \left(\frac{4}{7} + \frac{2}{3}\right) =$

(b) $\left(1\frac{1}{2} + \frac{3}{4}\right) \div \left(\frac{3}{8} - \frac{1}{7}\right) =$

(c) $1\frac{3}{4} \div \left(3\frac{1}{2} + 2\frac{1}{4}\right) =$

(d) $\left(1\frac{1}{2} \times 2\frac{1}{4}\right) + \left(3\frac{1}{8} \div \frac{2}{7}\right) =$

Solutions

1. (a) $\frac{3}{8}\times 1\frac{4}{7}=\frac{3}{8}\times\frac{11}{7}=\frac{33}{56}$

(b) $\frac{3}{6}+\frac{2}{7}=\frac{21}{42}+\frac{12}{42}=\frac{33}{42}=\frac{11}{14}$

(c) $1\frac{4}{5}-1\frac{1}{3}=1\frac{12}{15}-1\frac{5}{15}=\frac{7}{15}$

(d) $\frac{8}{17}\div 1\frac{1}{2}=\frac{8}{17}\div\frac{3}{2}=\frac{8}{17}\times\frac{2}{3}=\frac{16}{51}$

(e) $\frac{3}{8}+\frac{2}{7}=\frac{21}{56}+\frac{16}{56}=\frac{37}{56}$

(f) $5\frac{1}{2}\times 3\frac{1}{4}=\frac{11}{2}\times\frac{13}{4}=\frac{143}{8}=17\frac{7}{8}$

2. (a) $1\frac{1}{5}\times\left(\frac{4}{7}+\frac{2}{3}\right)=\frac{6}{5}\left(\frac{12}{21}+\frac{14}{21}\right)=\frac{6}{5}\times\frac{26}{21}=\frac{156}{105}=1\frac{17}{35}$

(b) $\left(1\frac{1}{2}+\frac{3}{4}\right)\div\left(\frac{3}{8}-\frac{1}{7}\right)=\left(\frac{6}{4}+\frac{3}{4}\right)\div\left(\frac{21}{56}-\frac{8}{56}\right)=\frac{9}{4}\div\frac{56}{13}=\frac{504}{52}=9\frac{9}{13}$

(c) $1\frac{3}{4}\div\left(3\frac{1}{2}+2\frac{1}{4}\right)=\frac{7}{4}\div\left(\frac{14}{4}+\frac{9}{4}\right)=\frac{7}{4}\div\frac{23}{4}=\frac{7}{4}\times\frac{4}{23}=\frac{28}{92}=\frac{7}{23}$

(d) $\left(1\frac{1}{2}\times 2\frac{1}{4}\right)+\left(3\frac{1}{8}\div\frac{2}{7}\right)=\left(\frac{3}{2}\times\frac{9}{4}\right)+\left(\frac{25}{8}\times\frac{7}{2}\right)=\frac{27}{8}+\frac{175}{16}=\frac{54}{16}+\frac{175}{16}=\frac{229}{16}=14\frac{5}{16}$